3. Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant using triple integration.

Sol

Gn: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1).

$vol = \iiint dz\, dy\, dx$.

To find x limit put $y=0$ and $z=0$ we get,

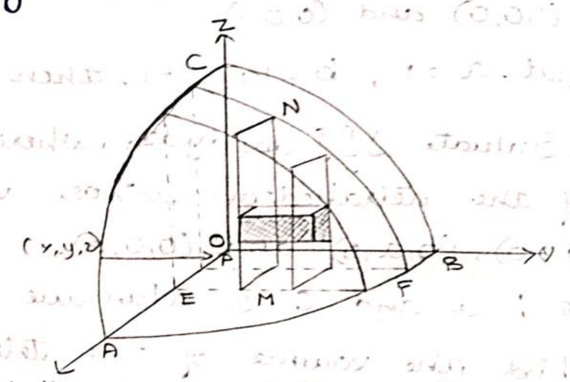
(1) $\Rightarrow \frac{x^2}{a^2}=1$

$x^2=a^2$

$x=\pm a$

(i.e.,) $x=0$ to $x=a$.

[$\because$ First octant area]



To find y limit put $z=0$ we get [surface integral]

(1) $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$

$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

(i.e.,) $y=0$ to $y=b\sqrt{1-\frac{x^2}{a^2}}$ [$\because$ first octant area]

To find z limit [Volume integral]

(1) $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$;

$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$; $z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

$$z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

(i.e.,) $z=0$ to $z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ [$\because$ first octant area]

$$\text{Volume} = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx = \int_0^a\int_0^{b\sqrt{1-x^2/a^2}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\,dy\,dx$$

Formula: $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$$= \frac{c}{b}\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\left(\sqrt{b^2(1-x^2/a^2)}\right)^2-y^2}\,dy\,dx$$

$$= \frac{c}{b}\int_0^a \left[\frac{y\sqrt{b^2(1-x^2/a^2)-y^2}}{2}+\frac{b^2(1-x^2/a^2)}{2}\sin^{-1}\left[\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right]\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{c}{b}\int_0^a \left(0+\frac{b^2(1-x^2/a^2)}{2}\right)\left(\frac{\pi}{2}\right)dx.$$

$$= \frac{c}{b} \int_0^a \left(0 + \frac{b^2 (1 - x^2/a^2)}{2} \left(\frac{\pi}{2}\right) \right) dx .$$

$$= \frac{\pi bc}{4} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx \quad = \frac{\pi bc}{4} \left[x - \frac{1}{a^2} \frac{x^3}{3} \right]_0^a$$

$$= \frac{\pi bc}{4} \left(a - \frac{a}{3} \right) \quad = \frac{\pi abc}{4} \left(1 - \frac{1}{3} \right)$$

$$= \frac{\pi abc}{4_2} \left(\frac{2}{3} \right)$$

$$= \frac{\pi abc}{6} \text{ cubic units}$$

Note: The volume of the whole ellipsoid

$$= 8 \left(\frac{\pi abc}{6} \right) = \frac{4}{3} \pi abc$$