UNIT - 4.

FUNCTIONS OF SEVERAL VARIABLE.

Differencial Equation :

* ODE
* PDE. $\longrightarrow$ Partial Differential Equation

PARTIAL DIFFERENTATION :

Let $U = f(x,y)$ be a function of two independent variable.

Differentiate 'U' w.r.t 'x', containing 'y' as constant is known as partial differential coefficient of 'u' w.r.t x. It is denoted by

$$\frac{\partial u}{\partial x}$$

Similarly if we differentiate U w.r.t 'y' considering x as constant is known as partial differential coefficient of U with respective to y. It is denoted by $\frac{\partial u}{\partial y}$.

$$\frac{\partial r}{\partial z} = \frac{z-c}{r}$$

Again diff w.r.t. 'z'

$$\frac{\partial^2 r}{\partial z^2} = \frac{r - (z-c)\left(\frac{\partial r}{\partial z}\right)}{r^2}$$

$$= \frac{r - (z-c)\frac{(z-c)}{r}}{r^2} = \frac{\frac{r^2 - (z-c)^2}{r}}{r^2}$$

$$= \frac{r^2 - (z-c)^2}{r^3} = \frac{1}{r} - \frac{(z-c)^2}{r^3} \quad \text{---} (4)$$

Adding (2) + (3) + (4)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}$$

$$= \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} - \frac{(y-b)^2}{r^3} + \frac{1}{r} - \frac{(z-b)^2}{r^3}$$

$$= \frac{3}{r} - \frac{(x-a)^2}{r^3} - \frac{(y-b)^2}{r^3} - \frac{(z-b)^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r^3}\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]$$

from (1)

$$= \frac{3}{r} - \frac{1}{r^3}(r^2)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}.$$ $\therefore$ Hence proved.

$$= \frac{r^2}{r^3} - \frac{(x-a)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} \quad \text{——— (2)}$$

Diff (1) w.r.t 'y'

$$2(y-b) = 2r \frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{2(y-b)}{2r}$$

$$\frac{\partial r}{\partial y} = \frac{(y-b)}{r}$$

Again diff w.r.t 'y'

$$\frac{\partial^2 r}{\partial y^2} = \frac{r - (y-b)\frac{\partial r}{\partial y}}{r^2}$$

$$= \frac{r - (y-b)\left(\frac{y-b}{r}\right)}{r^2} = \frac{\frac{r^2 - (y-b)^2}{r^2}}{r}$$

$$= \frac{r^2 - (y-b)^2}{r^3} = \frac{r^2}{r^3} - \frac{(y-b)^2}{r^3}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-b)^2}{r^3} \quad \text{——— (3)}$$

Diff (1) w.r.t. 'z'

$$2(z-c) = 2r \frac{\partial r}{\partial z}$$

$$\frac{\partial r}{\partial z} = \frac{2(z-c)}{2r}$$

$$\frac{\partial u}{\partial x} = e^y + ye^x \qquad d(x) = 1$$

$$d(e^x) = e^x$$

$$\frac{du}{\partial y} = xe^y + e^x$$

3. If $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

Given :-

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \quad \text{—— (1)}$$

Diff (1) w.r.t. 'x'

$$2(x-a) = 2r \frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{2(x-a)}{2r}$$

$$\frac{\partial r}{\partial x} = \frac{(x-a)}{r}$$

$u = x-a \quad u' = 1$

$v = r \quad v' = \frac{\partial r}{\partial x}$

$\frac{u}{v} = \frac{uv' - vu'}{v^2}$

Again Diff w.r.t. x

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - (x-a)\left(\frac{x-a}{r}\right)}{r^2}$$

$$= \frac{\frac{r^2 - (x-a)^2}{r}}{r^2} = \frac{r^2 - (x-a)^2}{r^3}$$

1. Find the 1^{st} and 2^{nd} derivative of $u = x^3 + y^3 - 3axy$.

Given :-

$$u = x^3 + y^3 - 3axy$$

$d(x^n) = nx^{n-1}$

$d(x) = 1$

I^{st} order.

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax$$

II^{nd} order

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x^2}\right) = \frac{\partial}{\partial x}(3x^2 - 3ay)$$

$$= 6x$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(3y^2 - 3ax)$$

$$= 6y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\cancel{(3xy^2 < 3ay}\ (3\cancel{y}^2 - 3a\cancel{x})$$

$$= -3a.$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 - 3ay)$$

$$= -3a.$$

2. Find the first order derivation of

$$u = xe^y + ye^x$$

Given : $u = xe^y + ye^x$

4. If $z = f(x+ct) + g(x-ct)$ then prove $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

Soln:-

Given:- $z = f(x+ct) + g(x-ct)$ ——(1)

$$\frac{\partial z}{\partial t} = f'(x+ct)c + g'(x-ct) -c$$

$$\frac{\partial z}{\partial t} = cf'(x+ct) - cg'(x-ct)$$

Again differentiating.

$$\frac{\partial^2 z}{\partial t^2} = cf''(x+ct)c - cg''(x-ct)(-c)$$

$$= c^2 f''(x+ct) + c^2 g''(x-ct) \quad \text{——(2)}$$

Diff (1) w.r.t. x

$$\frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct) \quad \text{——(3)}$$

From (2)

$$\frac{\partial^2 z}{\partial t^2} = c^2 [f''(x+ct) + g''(x-ct)]$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Hence proved.

5. [SNS logo] ...ify that $U_{xy} = U_{yx}$, $u = \tan^{-1}\left(\frac{x}{y}\right)$

Sol:

Given: $U = \tan^{-1}\left(\frac{x}{y}\right)$

$d(\tan^{-1}x) = \frac{1}{1+x^2}$

$$U_x = \frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$d(\tan^{-1}) = \frac{1}{1+x^2}$

$$= \frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= \frac{y^2}{y^2+x^2}\left(\frac{1}{y}\right)$$

$$= \frac{y}{y^2+x^2}$$

$$U_x = \frac{y}{x^2+y^2} \quad \text{——— (1)}$$

$d\left(\frac{x}{y}\right) = xy^{-1}$

$= -1xy^{-2}$

$= -\frac{x}{y^2}$

$$U_y = \frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{y^2}{x^2+y^2}\left(\frac{-x}{y^2}\right)$$

$$= -\frac{x}{x^2+y^2} \quad \text{——— (2)}$$

$$U_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) \qquad \partial\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2}$$

$$= -\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \qquad =$$

$$= -\left[\frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{-x^2+y^2}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{—— (3)}$$

$$U_{yx} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2(1)-2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{—— (4)}$$

From (3) and (4)

$$U_{xy} = U_{yx}$$

Hence proved.

6. If $U = e^{xy}$ prove $U_{xx} + U_{yy} = \frac{1}{U}[U_x^2 + U_y^2]$

Given $U = e^{xy}$

$$U_x = \frac{\partial U}{\partial x} = e^{xy} \cdot y$$

$$U_{xx} = \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}(y \cdot e^{xy})$$

$$= y^2 \cdot e^{xy} \quad \text{---} (1)$$

$$U_y = \frac{\partial U}{\partial y} = e^{xy} \cdot x.$$

$$U_{yy} = \frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial}{\partial y}(x e^{xy})$$

$$= x^2 e^{xy} \quad \text{---} (2)$$

Adding (1) + (2)

$$U_{xx} + U_{yy} = y^2 e^{xy} + x^2 e^{xy}$$

$$= e^{xy} [x^2 + y^2]$$

$$= U (x^2 + y^2) \quad \text{---} (3)$$

$$\frac{1}{U}[U_x^2 + U_y^2] = \frac{1}{e^{xy}}\left[(y e^{xy})^2 + (x e^{xy})^2\right]$$

$$= \frac{1}{e^{xy}}\left[y^2 e^{2xy} + x^2 e^{2xy}\right]$$

$$= \frac{e^{2xy}}{e^{xy}}[x^2 + y^2]$$

$$= e^{xy}[x^2 + y^2]$$

$$= U[x^2 + y^2] \quad \text{---} (4)$$

From (3) & (4) $U_{xx} + U_{yy} = \frac{1}{U}[U_x^2 + U_y^2]$

Hence proved.