

# Fundamentals of the Analysis of Algorithm Efficiency

- Analysis Framework
- Asymptotic Notations and its properties
- Mathematical analysis of Non - Recursive algorithms
- Mathematical analysis of Recursive algorithms



# Mathematical analysis of Non - Recursive algorithms

- Analysis framework – systematic – analyze the time efficiency of non-recursive algorithm
- **Example 1: Finding the largest value in a list of n numbers**

**ALGORITHM** *MaxElement(A[0..n - 1])*

*//Determines the value of the largest element in a given array*

*//Input: An array A[0..n - 1] of real numbers*

*//Output: The value of the largest element in A*

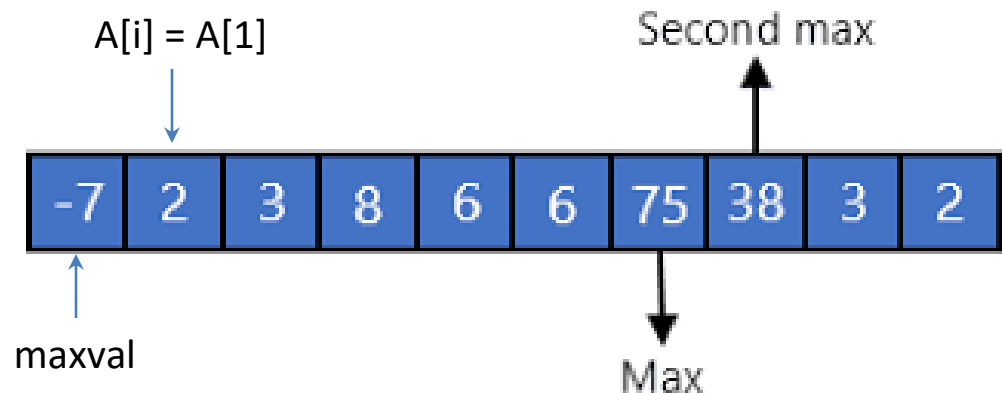
*maxval* ← A[0]

**for** *i* ← 1 **to** *n* - 1 **do**

**if** *A[i]* > *maxval*

*maxval* ← *A[i]*

**return** *maxval*



## Example 1: Finding the largest value in a list of n numbers

```
maxval ← A[0]
for i ← 1 to n - 1 do
    if A[i] > maxval
        maxval ← A[i]
return maxval
```

1	What is the problem size	$n$
2	What is the basic operation	Comparison in for loop
3	Count of basic operation	$C(n) = \sum_{i=1}^{n-1} 1 = n-1 \in \Theta(n)$
4	Depends on what efficiency? Worst/best/average	

# General Plan for Analyzing the Time Efficiency of Non recursive Algorithms

1. Decide on a parameter (or parameters) indicating **an input's size**.
2. Identify the algorithm's **basic operation**. (As a rule, it is located in the inner- most loop.)
3. Check whether the **number of times the basic operation is executed** depends only on the size of an input. If it also depends on some additional property, the **worst-case, average-case, and, if necessary, best-case efficiencies** have to be investigated separately.
4. **Set up a sum** expressing the number of times the algorithm's basic operation is executed.
5. Using **standard formulas** and rules of sum manipulation, either find a closed-form formula for the count or, at the very least, establish its order of growth.

# Formula for Sum Manipulation

$$\sum_{i=l}^u ca_i = c \sum_{i=l}^u a_i, \quad \text{(R1)}$$

$$\sum_{i=l}^u (a_i \pm b_i) = \sum_{i=l}^u a_i \pm \sum_{i=l}^u b_i, \quad \text{(R2)}$$

two summation formulas

$$\sum_{i=l}^u 1 = u - l + 1 \quad \text{where } l \leq u \text{ are some lower and upper integer limits, (S1)}$$

$$\sum_{i=0}^n i = \sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2 \in \Theta(n^2). \quad \text{(S2)}$$

## Example 2: Element Uniqueness Problem

<b>10</b>	<b>20</b>	<b>30</b>	<b>40</b>	<b>30</b>	<b>50</b>	<b>60</b>
1 <sup>st</sup>	2 <sup>nd</sup>	3 <sup>rd</sup>	4 <sup>th</sup>	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]

Here:  $n=7$ ,  $n-1 = 6$ ,  $n-2=5$

**ALGORITHM** *UniqueElements*(A[0..n - 1])

//Determines whether all the elements in a given array are distinct

//Input: An array A[0..n - 1]

//Output: Returns "true" if all the elements in A are distinct

// and "false" otherwise

**for**  $i \leftarrow 0$  **to**  $n - 2$  **do**

**for**  $j \leftarrow i + 1$  **to**  $n - 1$  **do**

**if**  $A[i] = A[j]$  **return false**

**return true**

## Example 2: Element Uniqueness Problem

1	What is the problem size	$n$
2	What is the basic operation	if statement (comparison)
3	Count of basic operation	$\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1:$
4	Depends on what efficiency? Worst/best/average <i><b>Worst case</b></i> – all the elements are different – all sequence of for loop <i><b>Best case</b></i> – 1 <sup>st</sup> and 2 <sup>nd</sup> element are same – comes out of loop	
5	Summation	

$$\begin{aligned}
 C_{\text{worst}}(n) &= \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i) \\
 &= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}
 \end{aligned}$$

## Example 3: Sum of n numbers

### Program:

```
count = 0;
for (i=1;i<=n;i++)
    count=count+i;
return count;
```

### Example:

$n = 5$ . count = 0

$i=1$  □ count =  $0+1 = 1$

$i=2$  □ count =  $1+2 = 3$

$i=3$  □ count =  $3+3 = 6$

$i=4$  □ count =  $6+4 = 10$

$i=5$  □ count =  $10+5 = 15$

### Analysis of sum of n numbers:

1. Problem size?
2. Basic Operation ?
3. Count of basic operation ?
4. Worst / Best / Average case efficiency ?



# Example 4: to find the no of binary digits in a binary representation of a positive decimal integer

$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
32	16	8	4	2	1
					1
				1	0
				1	1
	1	0	0	0	0

Number	Binary representation
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
15	1111
16	10000

## ALGORITHM *Binary(n)*

//Input: A positive decimal integer  $n$

//Output: The number of binary digits in  $n$ 's binary representation

$count \leftarrow 1$

**while**  $n > 1$  **do**

$count \leftarrow count + 1$

$n \leftarrow \lfloor n/2 \rfloor$

**return**  $count$

Iteration	n value	Count
Initial	2	1
1 <sup>st</sup>		2
	$n = n/2 = 1$	

Iteration	n value	Count
Initial	3	1
1 <sup>st</sup>		2
	$n = n/2 = 1.5$	

Iteration	n value	Count
Initial	4	1
1 <sup>st</sup>		2
	$n = 4/2 = 2$	
2 <sup>nd</sup>		3
	$n = 2/2 = 1$	

Iteration	n value	Count
Initial	8	1
1 <sup>st</sup>		2
	$n = 8/2 = 4$	
2 <sup>nd</sup>		3
	$n = 4/2 = 2$	
3 <sup>rd</sup>		4
	$n = 2/2 = 1$	

## Example 4: Analysis

1	What is the problem size	$n$
2	What is the basic operation	Comparison in while loop
3	Count of basic operation	$C(n) = \sum_{i=1}^{\lg(n)+1} 1$
4	Depends on what efficiency? Worst/best/average	

## Example 5: Matrix Multiplication

```
ALGORITHM MatrixMultiplication( $A[0..n - 1, 0..n - 1]$ ,  
 $B[0..n - 1, 0..n - 1]$ )  
//Multiplies two square matrices of order  $n$  by the definition-based algorithm  
//Input: Two  $n \times n$  matrices  $A$  and  $B$   
//Output: Matrix  $C = AB$   
for  $i \leftarrow 0$  to  $n - 1$  do  
  for  $j \leftarrow 0$  to  $n - 1$  do  
     $C[i, j] \leftarrow 0.0$   
    for  $k \leftarrow 0$  to  $n - 1$  do  
       $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$   
return  $C$ 
```

## Example 5: Matrix Multiplication Analysis

1	What is the problem size	<i>Order of matrix</i>
2	What is the basic operation	Multiplication and addition
3	Count of basic operation	$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1.$
4	Depends on what efficiency? Worst/best/average	
5	Running time $T(n) = C_m M(n) + C_a A(n)$ $= C_m n^3 + C_a n^3$ $= (C_m + C_a) n^3$	

Write the program for the following output and do the analysis process

1

1 2

1 2 3

1 2 3 4

1 2 3 4 5