



# SNS COLLEGE OF TECHNOLOGY

Coimbatore-35  
An Autonomous Institution



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Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

## DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

### 23ECT212 – LINEAR CONTROL SYSTEMS

II YEAR/ IV SEMESTER

#### UNIT I – CONTROL SYSTEM MODELING

#### TOPIC 3- DIFFERENTIAL EQUATION



# OUTLINE



- **REVIEW ABOUT PREVIOUS CLASS**
- **WHAT IS A DIFFERENTIAL EQUATION & TYPES**
- **TYPES OF ODE**
- **FIRST ORDER ODE-FIRST ORDER LINEAR ODE**
- **BERNOULI EQUATIONS**
- **SECOND ORDER ODE**
- **ACTIVITY**
- INITIAL / BOUNDARY VALUE PROBLEMS
- **Higher Order Homogeneous Differential ODE**
- **APPLICATIONS OF ODE**
- **EXAMPLES OF PDE**
  - ❖ Laplace Equation
  - ❖ Heat Equation
  - ❖ Wave Equation

**SUMMARY**



# WHAT IS DIFFERENTIAL EQUATION? & TYPES



A Differential Equation is an equation containing the derivative of one or more dependent variables with respect to one or more independent variables.

- [1. Ordinary Differential Equations.](#)
- [2. Partial Differential Equations.](#)
- [3. Linear Differential Equations.](#)
- [4. Non-linear differential equations.](#)
- [5. Homogeneous Differential Equations.](#)
- [6. Non-homogenous Differential Equations](#)

For Example,

$$\frac{dy}{dx} = 2xy$$

$$x \frac{dy}{dx} = y - 1$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} + 2 = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2 \frac{\partial u}{\partial t}$$



# TYPES OF DIFFERENTIAL EQUATION



## ODE (ORDINARY DIFFERENTIAL EQUATION):

An equation contains only ordinary derivatives of one or more dependent variables of a single independent variable.

For Example,

$$dy/dx + 5y = e^x,$$

$$(dx/dt) + (dy/dt) = 2x + y$$

## PDE (PARTIAL DIFFERENTIAL EQUATION):

An equation contains partial derivatives of one or more dependent variables of two or more independent variables.

For Example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2 \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$



# TYPES OF ODE



## ➤ FIRST ORDER ODE

- FIRST ORDER LINEAR ODE
- EXACT EQUATION
- NON-LINEAR FIRST ORDER ODE
- SEPERABLE EQUATION
- BERNOULLI DIFFERENTIAL EQUATION

## ➤ SECOND ORDER ODE

- LINEAR SECOND ORDER ODE
- HOMOGENEOUS SECOND ORDER ODE
- INITIAL AND BOUNDARY VALUE PROBLEMS
- NON-LINEAR SECOND ORDER ODE
- NON-HOMOGENEOUS SECOND ORDER ODE

## ➤ HIGHER ORDER ODE

- LINEAR NTH ORDER ODE
- HOMOGENEOUS EQUATION
- NON-HOMOGENEOUS EQUATION



# FIRST ORDER ODE

- A first order differential equation is an equation involving the unknown function  $y$ , its derivative  $y'$  and the variable  $x$ . We will only talk about explicit differential equations.

$$y' = f(x, y)$$

- General Form,

$$\frac{dy}{dx} = F(x, y),$$

- For Example,

$$\frac{dy}{dx} = 2x + 3$$



# FIRST ORDER LINEAR ODE

A **first order linear differential equation** has the following form:

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is given by

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

Where

called the **integrating factor**. If an initial condition is given, use it to find the constant  $C$ .

$$u(x) = \exp\left(\int p(x)dx\right)$$



# EXACT EQUATION



- Let a first order ordinary differential equation be expressible in this form:  
$$M(x,y)+N(x,y)dy/dx=0$$
such that M and N are *not* homogeneous functions of the same degree.

- However, suppose there happens to exist a function  $f(x,y)$  such that:

$$\partial f/\partial x=M, \quad \partial f/\partial y=N$$

such that the second partial derivatives of  $f$  exist and are continuous.

- Then the expression  $Mdx+Ndy$  is called an **exact differential**, and the differential equation is called an **exact differential equation**.





# SEPARABLE DIFFERENTIAL EQUATION



- A separable differential equation is any differential equation that we can write in the following form.

$$N(y) \frac{dy}{dx} = M(x)$$

- Note that in order for a differential equation to be separable all the  $y$ 's in the differential equation must be multiplied by the derivative and all the  $x$ 's in the differential equation must be on the other side of the equal sign.

$$N(y) dy = M(x) dx$$



# BERNOULI EQUATIONS

$$y' + p(x)y = q(x)y^n$$

- where  $p(x)$  and  $q(x)$  are continuous functions on the interval we're working on and  $n$  is a real number. Differential equations in this form are called **Bernoulli Equations**.
- If  $n=0$  or  $n=1$  then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of  $n$  other than these two.
- In order to solve these we'll first divide the differential equation by  $y^n$  to get,
- We are now going to use the substitution to convert this into a differential equation in terms of  $v$ . As we'll see this will lead to a differential equation that we can solve.

$$y^{-n} y' + p(x) y^{1-n} = q(x)$$

$$v = (1-n) y^{-n} y'$$



# SECOND ORDER ODE

The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t)$$

Initially we will make our life easier by looking at differential equations with  $g(t) = 0$ .

When  $g(t) = 0$  -- **Differential Equation Homogeneous** and when **---****Differential Equation Non- Homogeneous.**

$$g(t) \neq 0$$



# SECOND ORDER ODE...ACTIVITY



- how to go about solving a constant coefficient, homogeneous, linear, second order differential equation.
- Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

- For Example,

$$y'' - 9y = 0$$

- CONNECTIONS...START WITH THE WORD .....



# INITIAL / BOUNDARY VALUE PROBLEMS



- conditions specified at the extremes ("boundaries") of the independent variable in the equation .
- all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial" value).
  - For example, if the independent variable is time over the domain  $[0,1]$ , a boundary value problem would specify values for at both and , whereas an initial value problem would specify a value of and at time .
- EXAMPLE:
  - Finding the temperature at all points of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem.
  - If the problem is dependent on both space and time, one could specify the value of the problem at a given point for all time the data or at a given time for all space.
  - Concretely, an example of a boundary value (in one spatial dimension) is the problem



# INITIAL / BOUNDARY VALUE PROBLEMS



- To solve for the unknown function  $y(x)$  with the boundary conditions

$$y(0) = 0, y(\pi/2) = 2.$$

- Without the boundary conditions, the general solution to this equation is

$$y(x) = A \sin(x) + B \cos(x).$$

- From the boundary condition one obtains  $y(0) = 0$   
$$0 = A \cdot 0 + B \cdot 1$$

- which implies that  $B = 0$ . From the boundary condition  $y(\pi/2) = 2$  one finds

$$2 = A \cdot 1$$

- and so  $A = 2$ . One sees that imposing boundary conditions allowed one to determine a unique solution, which in this case is

$$y(x) = 2 \sin(x).$$



# Higher Order Homogeneous Differential ODE



- For Example,

$$y^{(3)} - 5y'' - 22y' + 56y = 0 \quad y(0) = 1 \quad y'(0) = -2 \quad y''(0) = -4$$

- The above equation is an example of **Higher Order Homogeneous Differential ODE** with initial conditions.

$$y^{(3)} - 12y'' + 48y' - 64y = 12 - 32e^{-8t} + 2e^{4t}$$

- **Similarly**, the above equation is an **Higher Order Non-Homogeneous Differential ODE** with coefficients.



# APPLICATIONS OF ODE



## ❖ **MODELLING WITH FIRST-ORDER EQUATIONS**

- Newton's Law of Cooling
- Electrical Circuits

## ❖ **MODELLING FREE MECHANICAL OSCILLATIONS**

- No Damping
- Light Damping
- Heavy Damping

## ❖ **MODELLING FORCED MECHANICAL OSCILLATIONS**

## ❖ **COMPUTER EXERCISE OR ACTIVITY**





# LINEAR & NON- LINEAR PDE



A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2 u_{xx} + 1 u_{xt} + 3 u_{tt} + 4 u_x + \cos(2t) = 0$$

$$2 u_{xx} - 3 u_t + 4 u_x = 0$$

Examples of Nonlinear PDE

$$2 u + (u) + 3 u = 0$$

$$u_{xx} + 2 u_{xt} + 3 u_t = 0$$

$$2 u_{xx} + 2 u_{xt} u_t + 3 u_t = 0$$



# EXAMPLES OF PDE

PDEs are used to model many systems in many different fields of science and engineering.

Important Examples:

- ❖ Laplace Equation
- ❖ Heat Equation
- ❖ Wave Equation



# LAPLACE EQUATION



- Laplace Equation is used to describe the steady state distribution of heat in a body.
- Also used to describe the steady state distribution of electrical charge in a body.

$$\frac{\partial^2 u(x, y, z)}{\partial x^2} + \frac{\partial^2 u(x, y, z)}{\partial y^2} + \frac{\partial^2 u(x, y, z)}{\partial z^2} = 0$$



# HEAT EQUATION



- The function  $u(x,y,z,t)$  is used to represent the temperature at time  $t$  in a physical body at a point with coordinates  $(x,y,z)$
- $\alpha$  is the thermal diffusivity. It is sufficient to consider the case  $\alpha = 1$ .

$$\frac{\partial u(x, y, z, t)}{\partial t} = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$



# WAVE EQUATION



- The function  $u(x,y,z,t)$  is used to represent the displacement at time  $t$  of a particle whose position at rest is  $(x,y,z)$  .
- The constant  $c$  represents the propagation speed of the wave.

$$\frac{\partial^2 u(x, y, z, t)}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$



# SUMMARY

