Curl of a vector:

Let Circulation of a vector field A around a closed path L as $\oint_L \vec{A} \cdot d\vec{l}$.

The curl of $\vec{A}$ is an axial or rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero & whose direction is the normal direction of the area when area is oriented to make the circulation maximum.

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \left(\lim_{\Delta s \to 0} \frac{\oint_L A \cdot dl}{\Delta s} \right)_{max} \hat{a}_n$$

where Δs is area bounded by the curve L

$\hat{a}_n$ is the unit vector normal to the surface ΔS

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

In cylindrical,

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

In spherical,

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \, \hat{a}_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & rA_\theta & r\sin\theta \, A_\phi \end{vmatrix}$$

Properties of curl

* The curl of a vector field is another vector field.

* The divergence of the curl of a vector field vanishes.

i.e. $\nabla \cdot (\nabla \times A) = 0$

* The curl of the gradient of a scalar field vanishes.

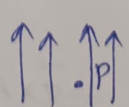
i.e. $\nabla \times \nabla V = 0$

Physical significance:

Curl provides the maximum value of the circulation of the field per unit area (or circulation density).

Indicates the direction along which this maximum value occurs.

Curl at P point out of the page



Curl at P is zero

Stokes's theorem:

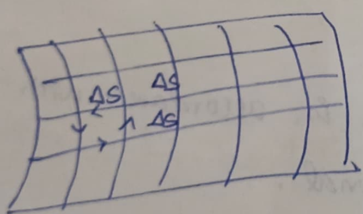
States that the circulation of a vector field $\vec{A}$ around a (closed) path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L.

$$\oint_L \vec{A} \cdot \vec{dl} = \int_S (\vec{\nabla} \times \vec{A}) \cdot \vec{ds}$$

Proof:

Consider any surface with area S. It is subdivided into different areas Δs. By ampere's law,

$$\oint \vec{A} \cdot \vec{dl} = \oint_1 A \cdot dl + \oint_2 A \cdot dl + \ldots \rightarrow ①$$



from the definition of curl.

$$\underset{\Delta s \to 0}{Lt} \frac{\oint_L \vec{A} \cdot \vec{dl}}{\Delta s} \hat{a}_n = \nabla \times A$$

~~Sub ② in ①~~

$$\oint \vec{A} \cdot \vec{dl} = \underset{\Delta s \to 0}{Lt} (\nabla \times A) \cdot \Delta s$$

Sub ② in ①

$$\oint \vec{A}\cdot d\vec{l} = \underset{\Delta s_1 \to 0}{Lt} (\nabla \times A)\cdot \Delta s_1 + \underset{\Delta s_2 \to 0}{Lt} (\nabla \times A)\cdot \Delta s_2 + \underset{\Delta s_3 \to 0}{Lt} (\nabla \times A)\cdot \Delta s_3$$

$$\oint \vec{A}\cdot d\vec{l} = \oint_S (\nabla \times A)\cdot ds$$

Solenoidal & Irrotational

* A divergence less field is called as solenoidal field.

$\nabla \cdot \vec{A} = 0$

Eg: Magnetic field

$\nabla \cdot B = 0$ means the magnetic flux lines close upon themselves and that there are no magnetic sources or sinks ($B = \nabla \times A$)

* A curl-free vector field is called as an irrotational or conservative field.

Eg. Electrostatic field.

$\nabla \times E = 0$

We may classify vector fields in accordance with their being solenoidal and/or irrotational.

1. Solenoidal & irrotational if $\nabla \cdot F = 0$ and $\nabla \times F = 0$

Eg: Static electric field in a charge free region

2. Solenoidal but not irrotational if $\nabla \cdot F = 0$, $\nabla \times F \neq 0$

Eg: A steady magnetic field.

3. Irrotational but not! solenoidal if

$\nabla \times F = 0$ and $\nabla \cdot F \neq 0$

Eg: Static electric field in a charged region

4. Neither solenoidal nor irrotational if

$\nabla \cdot F \neq 0$ and $\nabla \times F \neq 0$

Eg: An electric field in a charge medium with a time-varying magnetic field

① Proof of $\nabla \cdot (\nabla \times A) = 0$

For any vector field $\vec{A}$, show explicitly that

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ i.e, the divergence of the curl of any vector field is zero.

Solution :

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \hat{a_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{a_y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{a_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \left(\frac{\partial}{\partial x}\hat{a_x} + \frac{\partial}{\partial y}\hat{a_y} + \frac{\partial}{\partial z}\hat{a_z}\right) \cdot (\vec{\nabla} \times \vec{A})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x \partial y} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

(2) Prove $\vec{\nabla} \times \vec{\nabla} V = 0$

Show that Curl of the gradient of any scalar field vanishes.

Solution:

$$\text{grad } V = \frac{\partial V}{\partial x}\hat{a_x} + \frac{\partial V}{\partial y}\hat{a_y} + \frac{\partial V}{\partial z}\hat{a_z}$$

$$\text{Curl} \cdot \text{grad } V = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \hat{a_x}\left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z}\right] - \hat{a_y}\left[\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right] +$$

$$\hat{a_z}\left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y}\right]$$

$$\boxed{(\vec{\nabla} \times \vec{\nabla} V) = 0}$$

* If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

Eg: Let a vector field E. If $\nabla \times E = 0$,

then $E = -\nabla V$.