SNS COLLEGE OF TECHNOLOGY

Coimbatore-28
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DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECT212 – CONTROL SYSTEMS

II YEAR/ IV SEMESTER

UNIT II – TIME RESPONSE ANALYSIS

TOPIC 5- IMPULSE AND STEP RESPONSE ANALYSIS OF SECOND ORDER SYSTEMS

OUTLINE

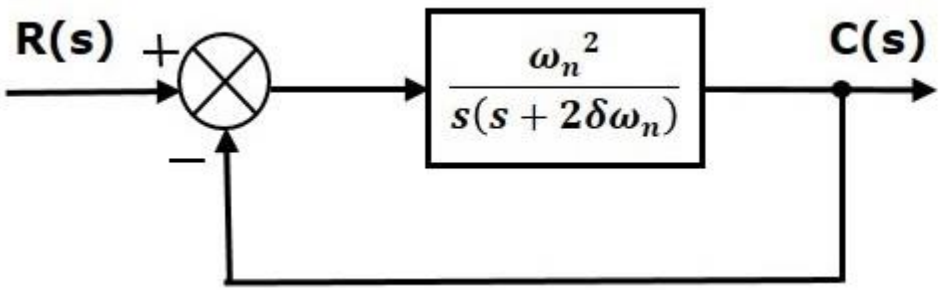
- REVIEW ABOUT PREVIOUS CLASS
- INTRODUCTION
- SECOND ORDER SYSTEM
- STEP RESPONSE OF SECOND ORDER SYSTEM
- ACTIVITY
- IMPULSE RESPONSE OF SECOND ORDER SYSTEM
- DAMPING RATIO
- 1.Undamped system, $\zeta = 0$
- 2. Under damped system, $0 < \zeta < 1$
- 3. Critically damped system, $\zeta = 1$
- 4. Over damped system, $\zeta > 1$
- SUMMARY

Second Order System

- the affect of location of poles and zeros on the transient response of 1st order systems already discussed
- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.
- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.
- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

INTRODUCTION

- Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\frac{\omega_n^2}{s(s+2\zeta\omega_n)}$ is connected with a unity negative feedback.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

INTRODUCTION

ω_n ⟶ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ ⟶ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow \zeta = 0.5$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

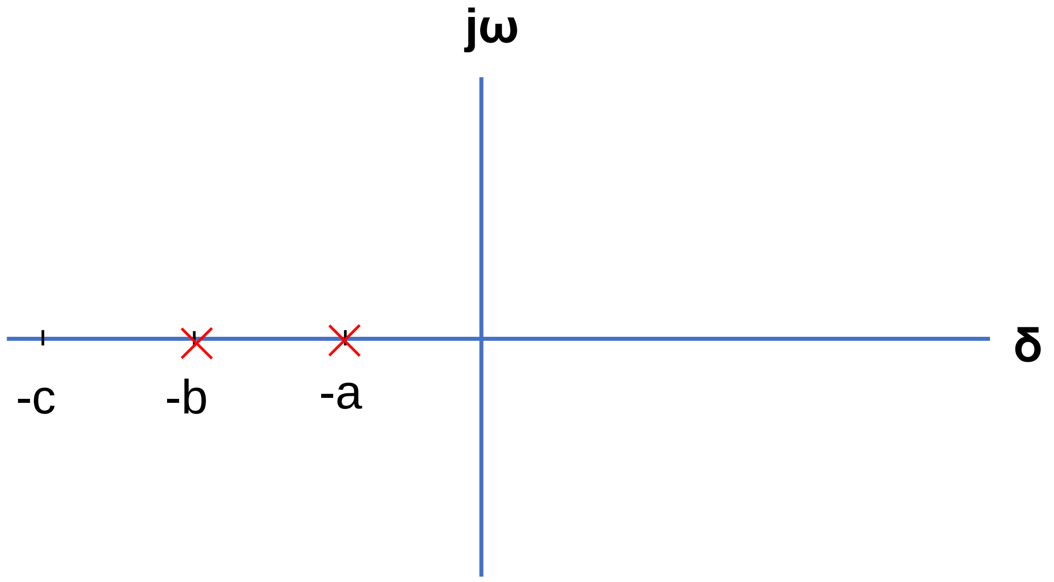
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories :

  1. Overdamped - when the system has two real distinct poles (ζ >1).

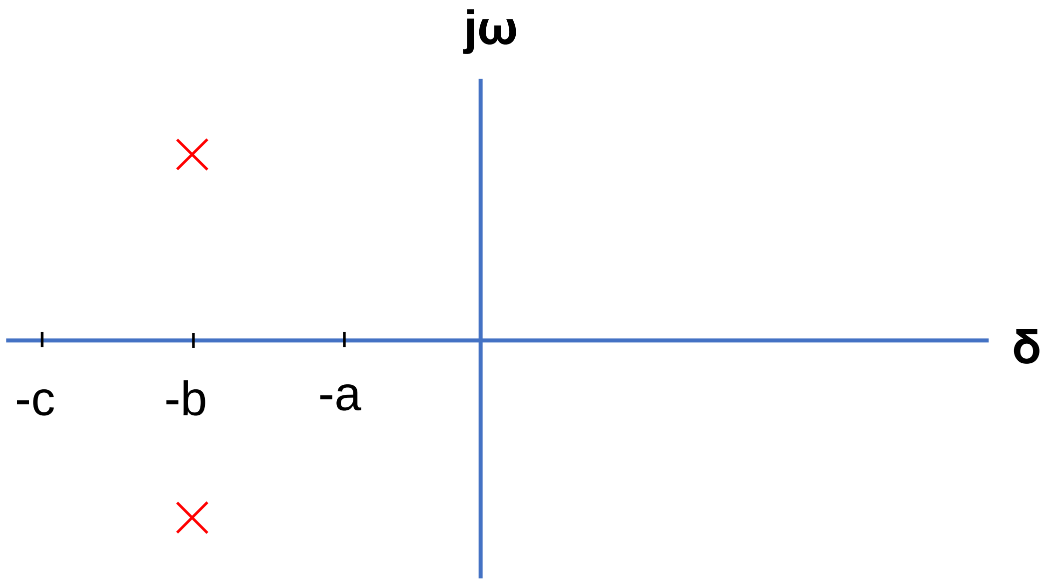


$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ, a second-order system can be set into one of the four categories

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)

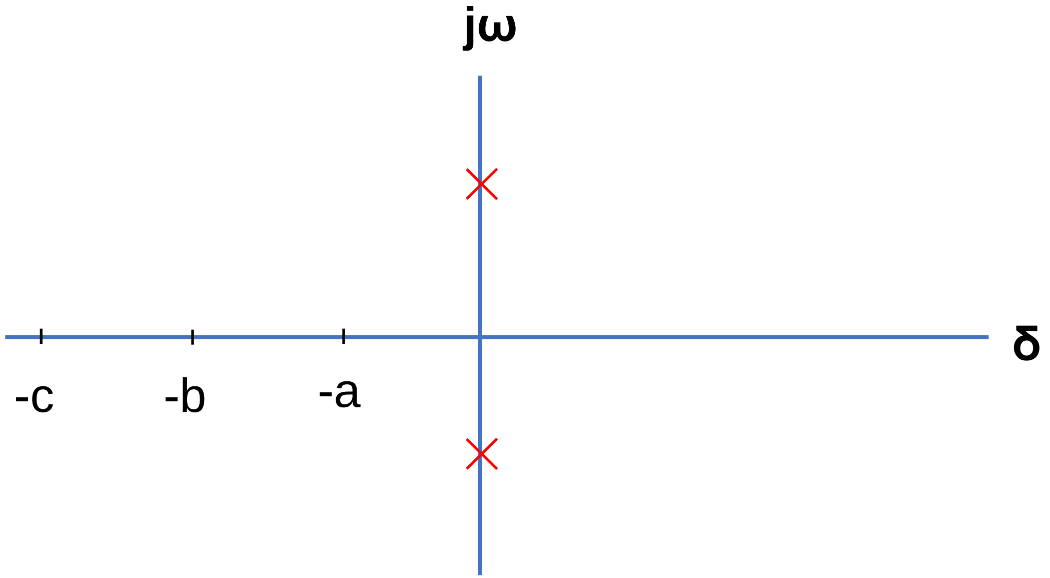


$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

3. *Undamped* - when the system has two imaginary poles (ζ = 0).

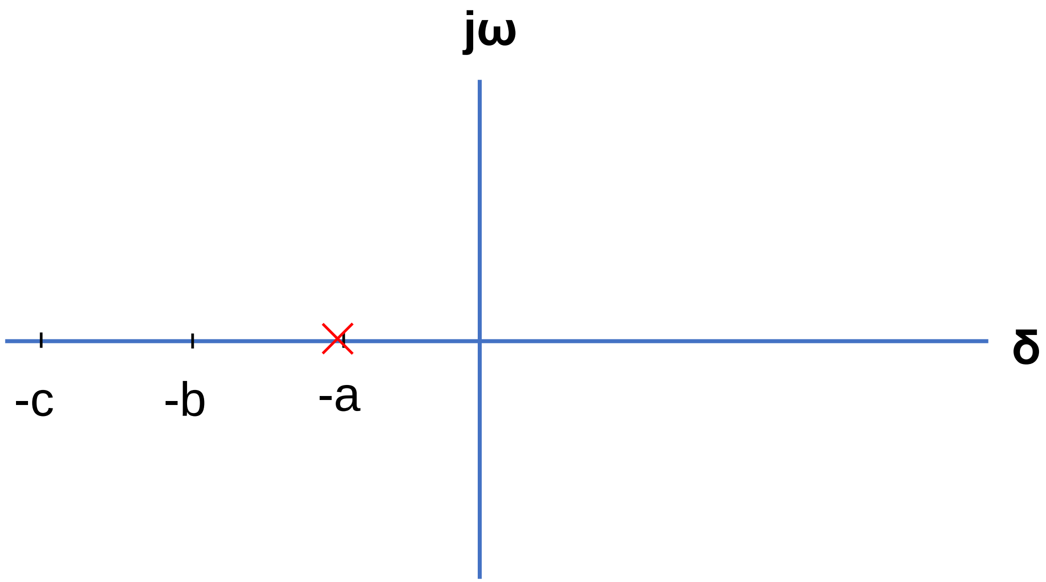


$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



SECOND ORDER SYSTEM

Transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of ‘s’ is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The two roots are imaginary when $\delta = 0$.
- The two roots are real and equal when $\delta = 1$.
- The two roots are real but not equal when $\delta > 1$.
- The two roots are complex conjugate when $0 < \delta < 1$.

We can write $C(s)$ equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

Where,
C(s) is the Laplace transform of the output signal, c(t)
R(s) is the Laplace transform of the input signal, r(t)
$\boldsymbol{\omega_n}$ is the natural frequency
δ is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain

- Take Laplace transform of the input signal, r(t)r(t).
- Consider the equation,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

- Substitute R(s)R(s) value in the above equation.
- Do partial fractions of C(s)C(s) if required.
- Apply inverse Laplace transform to C(s)C(s).

STEP RESPONSE OF SECOND ORDER SYSTEM

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s)=1 / s$$

We know the transfer function of the second order closed loop control system is,

STEP RESPONSE ...

the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

STEP RESPONSE…

Apply inverse Laplace transform on both the sides.

$$c(t)=(1-\cos(\omega nt))u(t)$$

unit step response of the second order system when /delta=0 will be a continuous time signal with constant amplitude and frequency.

STEP RESPONSE...

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

ACTIVITY

GROUP DISCUSSION

IMPULSE RESPONSE OF SECOND ORDER SYSTEM

The **impulse response** of the second order system can be obtained by using any one of these two methods.

- Follow the procedure involved while deriving step response by considering the value of R(s) as 1 instead of 1/s.
- Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	Impulse response for t ≥ 0
δ = 0	$\omega_n \sin(\omega_n t)$
δ = 1	$\omega_n^2 t e^{-\omega_n t}$
0 < δ < 1	$\left(\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t)$
δ > 1	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}}\right)\left(e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t}\right)$

DAMPING RATIO

- **Refer the first slide diagram for ref.**
- Depending on the value of damping ratio, second order system can be classified into:
  1. Undamped system, $\zeta = 0$
  2. Underdamped system, $0 < \zeta < 1$
  3. Critically damped system, $\zeta = 1$
  4. Overdamped system, $\zeta > 1$
- The characteristic equation is given by,

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

DAMPING RATIO...

- The roots of characteristic equation is given by,

$$s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The roots are imaginary when ζ = 0
- The roots are real and equal when ζ = 1
- The roots are real and unequal when ζ >1
- The roots are complex conjugate when 0 < ζ < 1

UNDAMPED SYSTEM

- Step Response of undamped second order system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- When $\zeta = 0$,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$c(t) = (1 - \cos(\omega_n t))$$

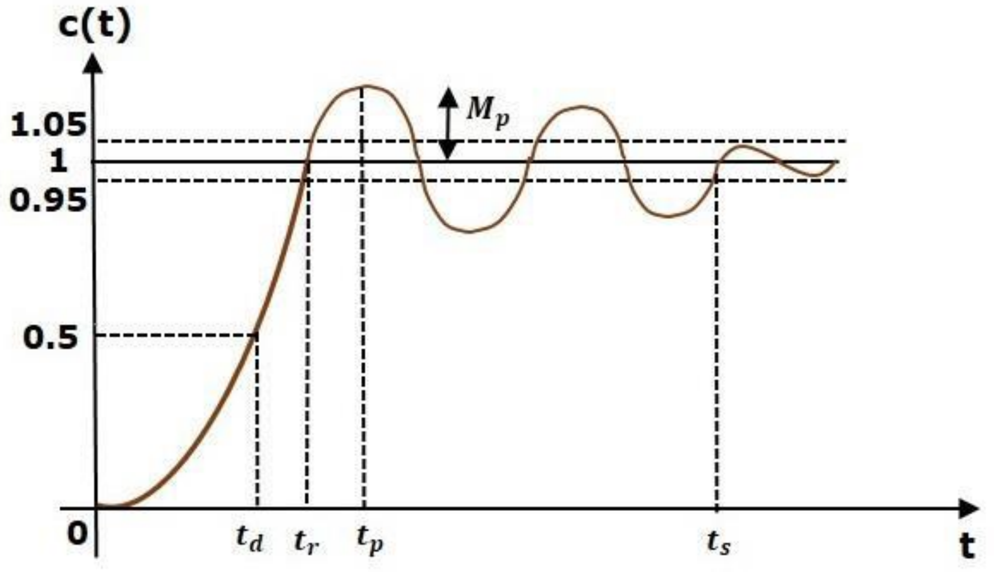
UNDERDAMPED SYSTEM

- Step response of a second order underdamped system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{A}{s} + \frac{(Bs + C)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right)$$



CRITICALLY DAMPED SYSTEM

- Step response of a second order critically damped system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

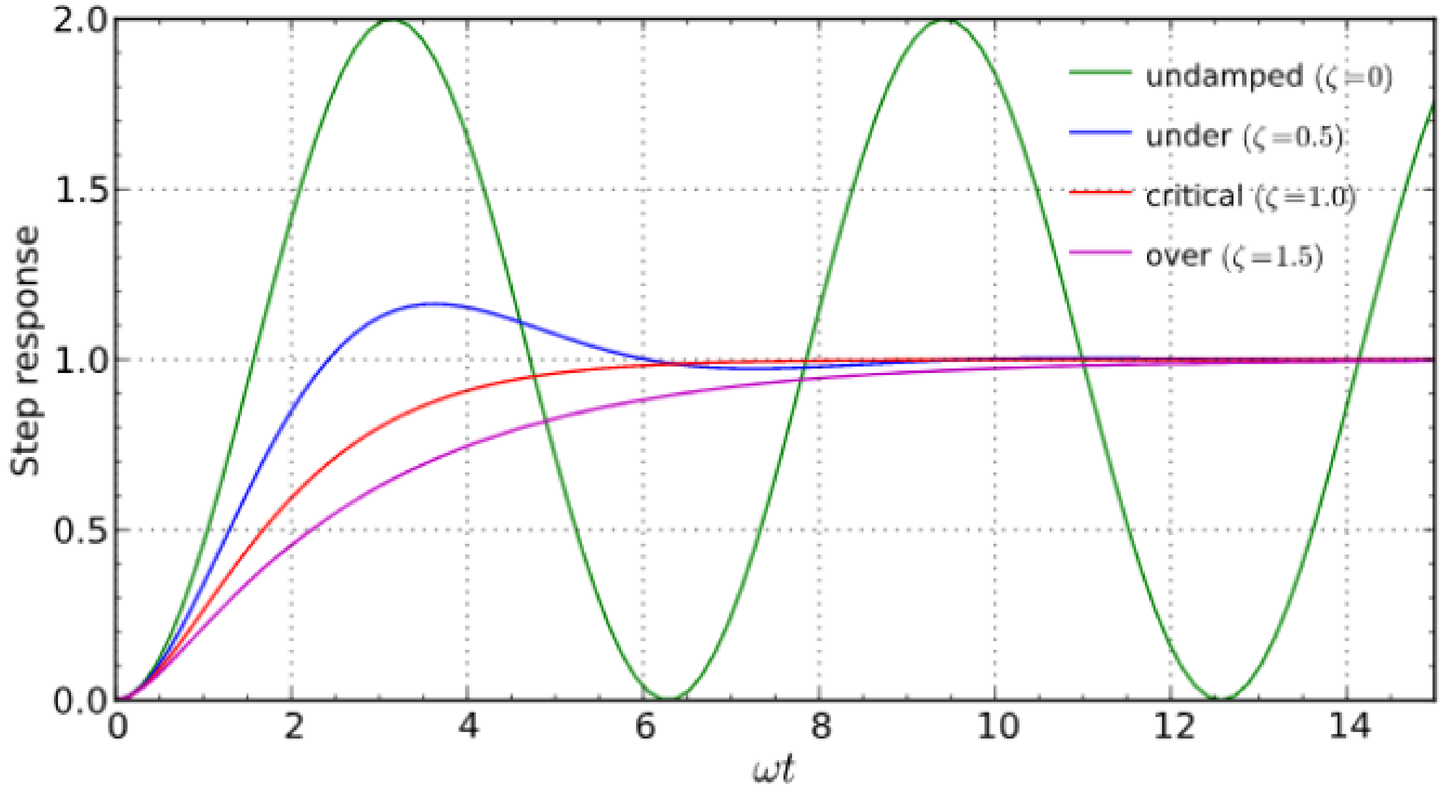
- When ζ = 1,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})$$

CRITICALLY DAMPED SYSTEM



SUMMARY