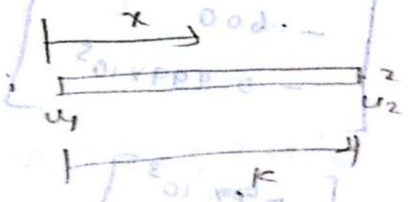
The Derivation of displacement function (u) and shape function (N) For one dimensional bar element based on global co-ordinate approach:-

consider a bar element with nodes 1 and 2 as shown in fig. u_1 and u_2 are the displacements at respective nodes. So u_1 and u_2 are considered as degree's of freedom of this bar element.



since the element has two degree's of freedom it will have two generalize co-ordinate.

$$u = a_0 + a_1 x \longrightarrow (1)$$

where, a_0 and a_1 are Global (or) generalize co-ordinates.

writing the equ (1) in matrix form

$$= \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \longrightarrow (2)$$

At node 1,

$u = u_1, \quad x = 0$

At node 2,

$u = u_2, \quad x = l$

Sub that values in equation ①.

$$u_1 = a_0, \longrightarrow ③$$

$$u_2 = a_0 + a_1 l \rightarrow ④.$$

Arrange the equ ③ + ④ in matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \rightarrow ⑤.$$

$\downarrow u^*$, $\downarrow C$, $\downarrow A$

u^* – degree's of freedom

C – conectivity matrix

A – Generalize (or) global co-ordinate matrix

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{(a_{11} a_{22} - a_{12} a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \cdot \frac{1}{(l - 0)} \times \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}$$

substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation ②.

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{bmatrix} l-x & x \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$\therefore N_1 = \frac{l-x}{l}$; $N_2 = \frac{x}{l}$

Displacement function $u = N_1 u_1 + N_2 u_2$

Shape function $N_1 = \frac{l-x}{l}$

Shape function $N_2 = x/l$

At node 1 : $x = 0$

$N_1 = 1$, $N_2 = 0$

At node 2 : $x = l$

$N_1 = 0$, $N_2 = 1$

The shape function will have a value equal to unity at the node to which it belongs and zero value at other node

Derivation of stiffness matrix for one dimensional bar Element :-

$$[K] = \int_v [B]^T [D] [B] \, dv$$

In one dimentional bar element.

Displacement function $u = N_1 u_1 + N_2 u_2$.

Where, $N_1 = \frac{l-x}{l}$, $N_2 = x/l$.

Strain displacement matrix [B]

$$[B] = \left[\frac{d(N_1)}{dx} \cdot \frac{d(N_2)}{dx} \right]$$

$$= \left[-1/l \quad 1/l \right]$$

$$[B]^T = \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}$$

[D] - stress strain relationship matrix. which is equal to young's modulus.

$$[K] = \int_0^l \begin{pmatrix} -1/l \\ 1/l \end{pmatrix} \times E \left[-1/l \quad 1/l \right] dv$$

$$[K] = \int_0^l \begin{bmatrix} +1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} E \, dv$$

$$dv = A\,dx.$$

$$[K] = AE \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} \int_0^l dx$$

$$= AE \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} [x]_0^l$$

$$= AE \begin{bmatrix} 1/l & -1/l \\ -1/l & 1/l \end{bmatrix}.$$

$$= \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The properties of stiffness matrix are satisfied

1) It is symmetric

2) The sum of elements in any column is equal to zero.