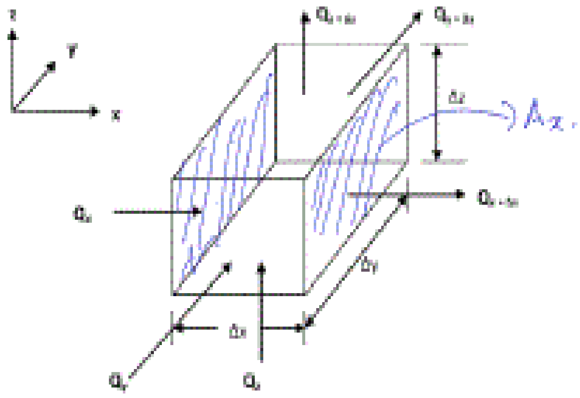
General 3D-Conduction equation in Cartesian coordinates

Energy balance

$$Q_{net} + Q_{gen} = \frac{\partial E}{\partial t}.$$



x-direction:

Heat entering the element $= Q_x \longrightarrow (1)$

Heat exiting the element $= Q_{(x+\Delta x)} \longrightarrow (2)$

From the taylor's series;

$$Q_{x+\Delta x} = Q_x + \frac{\partial Q_x}{\partial x}\Delta x + \underbrace{\frac{\partial^2 Q_x}{\partial x^2}\cdot\frac{\Delta x^2}{2!} + \frac{\partial^3 Q_x}{\partial x^3}\cdot\frac{\Delta x^3}{3!} \ldots}_{\text{Neglecting higher order terms}}\Big]$$

Net heat flow is given by;

$$Q_x - Q_{x+\Delta x} = Q_x - \left[Q_x + \frac{\partial Q_x}{\partial x}\cdot\Delta x\right].$$

$$= -\frac{\partial Q_x}{\partial x}\cdot\Delta x. \longrightarrow (3)$$

From fourier law $Q_x = -k_x A_x \frac{\partial T}{\partial x}$.

$$\therefore\ Q_x - Q_{x+\Delta x} = -\frac{\partial\left[-k_x A_x \partial T/\partial x\right]}{\partial x}.$$

$$= k_x \cdot \frac{\partial^2 T}{\partial x^2} \cdot \Delta x \cdot A_x$$

$$= k_x \cdot \frac{\partial^2 T}{\partial x^2} \cdot \underbrace{\Delta x \cdot \Delta y \cdot \Delta z}_{\text{Volume of the element.}} \rightarrow (4) \quad \because A_x = \Delta y \cdot \Delta z$$

Similar expressions can be obtained from 'y' & 'z' direction.

$$Q_y - Q_{y+\Delta y} = k_y \cdot \frac{\partial^2 T}{\partial y^2} \Delta x \Delta y \Delta z \rightarrow (5)$$

$$Q_z - Q_{z+\Delta z} = k_z \cdot \frac{\partial^2 T}{\partial z^2} \cdot \Delta x \cdot \Delta y \cdot \Delta z \longrightarrow (6)$$

Net heat flow in all three directions [x, y, z].

$$Q_{Net} = \left[k_x \cdot \frac{\partial^2 T}{\partial x^2} + k_y \cdot \frac{\partial^2 T}{\partial y^2} + k_z \cdot \frac{\partial^2 T}{\partial z^2}\right] \Delta x \, \Delta y \cdot \Delta z \rightarrow (7)$$

Heat generated from the volume.

$$Q_{gen} = q''' \cdot \Delta x \cdot \Delta y \cdot \Delta z \rightarrow (8)$$

$\because q'''$ = Heat produced per unit volume within the element.

From the first law of thermodynamics;

$$Q_{Net} + Q_{gen} = \text{Rate of change of energy transfer.}$$

$$\rho = \frac{m}{\Delta V} = \frac{m}{\Delta x \Delta y \Delta z} \qquad = \frac{\partial E}{\partial t} = \frac{m \cdot C_p \cdot \Delta T}{\partial t}$$

$$m = \rho \cdot \Delta x \Delta y \Delta z \qquad = \rho \cdot C_p \, \Delta x \Delta y \, \Delta z \cdot \frac{\Delta T}{\partial t} \rightarrow (9)$$

$$\therefore \left[k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} + q'''\right] \underbrace{\Delta x \cdot \Delta y \cdot \Delta z}_{\text{Volume}} = \rho C_p \frac{\Delta T}{\partial t} \underbrace{\Delta x \Delta y \Delta z}_{\text{Volume.}}$$

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} + q''' = \rho C_p \frac{\Delta T}{\partial t}$$

[OR]

$$\frac{\partial}{\partial x}\left[k_x \frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y \frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] + q''' = \rho C_p \frac{\Delta T}{\partial t}$$

(OR) Using vector operator ∇, $\rightarrow$ (10)

$$\nabla \cdot (k \cdot \nabla t) + q''' = \rho C_p \cdot \frac{\Delta T}{\partial t}$$

This is the general heat conduction equation for 'non-homogeneous material', 'self heating' and 'unsteady three-dimensional heat flow'. This equation establishes differential form of relationship between the time and space variation of temperature at any point of solid through which heat flow by conduction take place

Case-1: Isotropic material $k = k_x = k_y = k_z$.

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+q''' = \rho C_p \cdot \frac{\partial T}{\partial t}.$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q'''}{k} = \frac{\rho C_p}{k}\cdot\frac{\partial T}{\partial t}. \qquad \left[\because \alpha = \frac{k}{\rho C_p}\right].$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q'''}{k} = \frac{1}{\alpha}\cdot\frac{\partial T}{\partial t} \longrightarrow (11)$$

[Fourier-Biot equation]

Case-2: $q''' = 0$ [No heat generation and isotropic].

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = \frac{\rho C_p}{k}\cdot\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\cdot\frac{\partial T}{\partial t} \longrightarrow (12)$$

[Diffusion equation].

Case-3: $\frac{\partial T}{\partial t} = 0$ [Steady state and isotropic]

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q'''}{k} = 0. \longrightarrow (13)$$

[Poisson's equation].

Case-4: $q''' = 0$ & $\frac{\partial T}{\partial t} = 0$ [Steady state, no heat generation and isotropic]

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = 0 \longrightarrow (14)$$

[Laplace equation].