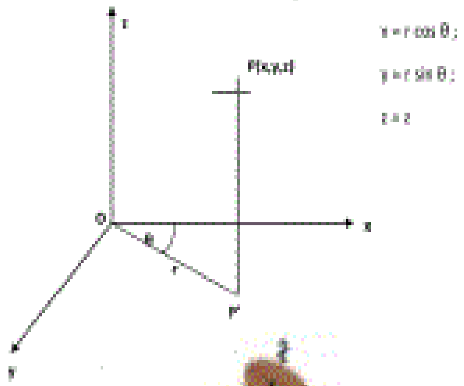
SNS COLLEGE OF TECHNOLOGY, COIMBATORE-35

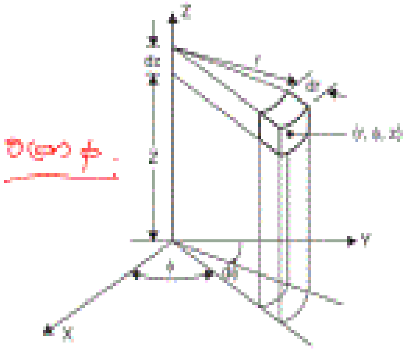
DEPARTMENT OF MECHANICAL ENGINEERING

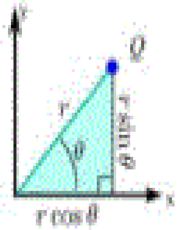
19MEB302/ Heat and Mass Transfer – **UNIT I - CONDUCTION**

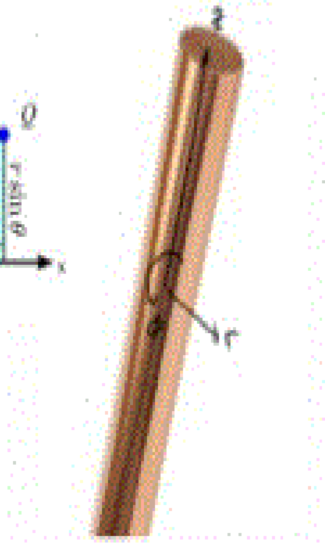
Topic - Cylindrical Coordinates

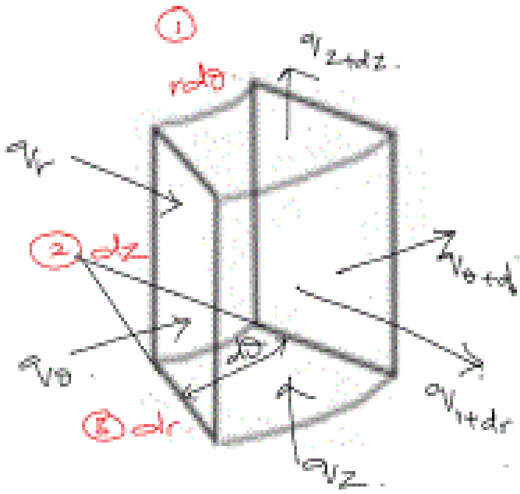
Three dimensional conduction equation in cylindrical coordinates [r, θ, z].











Cylindrical coordinates are r, θ, z.

Transformation of coordinates:

$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.

From chain rule,

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial r} \rightarrow ①$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\cdot\cos\theta + \frac{\partial T}{\partial y}\sin\theta$$

Multiplying both sides by $\cos\theta$.

$$\cos\theta \frac{\partial T}{\partial r} = \cos^2\theta \frac{\partial T}{\partial x} + \sin\theta . \cos\theta . \frac{\partial T}{\partial y} \rightarrow (2)$$

Similarly;

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x} . \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} . \frac{\partial y}{\partial \theta} \rightarrow (3)$$

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x}(-r\sin\theta) + \frac{\partial T}{\partial y}(r\cos\theta).$$

Multiply both sides by $\sin\theta / r$.

$$\therefore \frac{\sin\theta}{r} . \frac{\partial T}{\partial \theta} = -\sin^2\theta \frac{\partial T}{\partial x} + \sin\theta . \cos\theta . \frac{\partial T}{\partial y} . \rightarrow (4)$$

From the equation (2) and (4) we have,

$$\frac{\sin\theta}{r} . \frac{\partial T}{\partial \theta} = -\sin^2\theta \frac{\partial T}{\partial x} + \left[\cos\theta \frac{\partial T}{\partial r} - \cos^2\theta \frac{\partial T}{\partial x}\right]$$

$$= -\sin^2\theta \frac{\partial T}{\partial x} + \cos\theta \frac{\partial T}{\partial r} - \cos^2\theta \frac{\partial T}{\partial x}.$$

$$= -\frac{\partial T}{\partial x}\left[\sin^2\theta + \cos^2\theta\right] + \cos\theta \frac{\partial T}{\partial r}.$$

$$\frac{\sin\theta}{r} . \frac{\partial T}{\partial \theta} = -\frac{\partial T}{\partial x} + \cos\theta \frac{\partial T}{\partial x}.$$

$$\frac{\partial T}{\partial x} = \cos\theta \frac{\partial T}{\partial x} - \frac{\sin\theta}{r} . \frac{\partial T}{\partial \theta} \rightarrow (5)$$

Differentiating the above eqn w.r.to 'x', we have

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left[\cos\theta \cdot \frac{\partial T}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial T}{\partial \theta}\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \cos\theta \cdot \frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right) - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial x}\right)$$

Substituting for $\frac{\partial T}{\partial x}$ from. equation (5)

$$\frac{\partial^2 T}{\partial x^2} = \cos\theta \frac{\partial}{\partial r}\left[\cos\theta \cdot \frac{\partial T}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial T}{\partial \theta}\right] - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}\left[\cos\theta \cdot \frac{\partial T}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial T}{\partial \theta}\right].$$

$$= \cos^2\theta \cdot \frac{\partial^2 T}{\partial r^2} - \frac{\cos\theta \sin\theta}{r^2} \cdot \frac{\partial T}{\partial \theta} + \frac{\sin^2\theta}{r} \cdot \frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2}$$

$$+ \frac{\sin\theta \cos\theta}{r^2} \cdot \frac{\partial T}{\partial \theta} \rightarrow (6)$$

Similarly. we have for;

$$\frac{\partial^2 T}{\partial y^2} = \sin^2\theta \cdot \frac{\partial^2 T}{\partial r^2} + \frac{\cos^2\theta}{r} \cdot \frac{\partial T}{\partial r} - \frac{\cos\theta \cdot \sin\theta}{r^2} \cdot \frac{\partial T}{\partial \theta} + \frac{\cos^2\theta}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2}$$

$$+ \frac{\cos\theta \sin\theta}{r^2} \cdot \frac{\partial T}{\partial \theta} \rightarrow (7)$$

By adding equation (6) & (7) we get.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial r^2}\left(\sin^2\theta + \cos^2\theta\right) + \frac{1}{r} \cdot \frac{\partial T}{\partial r}\left(\sin^2\theta + \cos^2\theta\right)$$

$$+ \frac{1}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2}\left(\sin^2\theta + \cos^2\theta\right)$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2} \rightarrow (8)$$

Substituting in the 3D heat conduction of Cartesian coordinates.

$$\underbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{K} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t}.$$

$$\underbrace{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}} + \frac{1}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{K} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \rightarrow (9)$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{q'''}{K} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t}.$$

1) One dimensional case:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) + \frac{q'''}{K} = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \rightarrow (10)$$

2) 1-D with no heat generation

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha} \cdot \frac{\partial T}{\partial t} \rightarrow (11)$$

3) 1-D with steady state and no heat generation

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) = 0$$