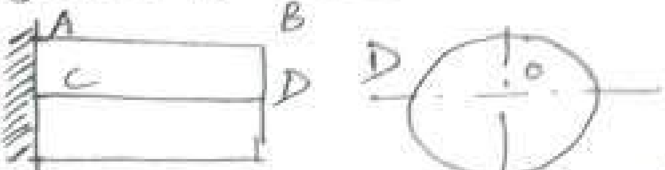
UNIT-3

TORSION OF SHAFTS & SPRINGS

A shaft is said to be torsion, when equal & opposit. torques are applied at two ends of the shaft. The Torque is equal to the product of force and radius of the shaft. Due to application of torque at 2 ends, the shaft is subjected to twisting moment This causes the shear stresses and shear strains in the materials of the shaft

Derivation of shear stress produced in a circular shaft subjected to torsion:



R = radius of shaft
L = length of the shaft
T = Torque applied

τ = shear stress induced at the surface of the shaft due to the torque (T)

ϕ = shear strain angle

Modulus of rigidity (C)

$$C = \frac{\text{shear stress induced}}{\text{shear strain produced}}$$

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$C = \tau / R\theta/L$$

Assumption made in Derivation of shear stress produced in the circular shaft

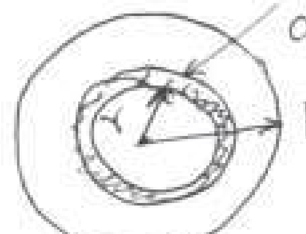
1. The material of the shaft is uniform throughout
2. The twist along the shaft is uniform.
3. The shaft of uniform cross section throughout.
4. Cross sections of the shaft is, which are plane before twist and remains plain after twist.
5. All radii which are straight before twist remain straight after twist

The maximum torque transmitted by solid circular shaft is obtained from maximum shear stress induced at the outer surface of solid shaft.

Maximum Torque transmitted by a solid circular shaft

The maximum torque is transmitted at the shaft. Consider a shaft to subjected to the torque

τ = maximum shear stress; R = radius of the shaft



Turning force on elementary circular ring = (Shear stress act on the ring) × Area of ring

$= q \times dA = \frac{\tau}{R} \times 2\pi r^2 dr$

Now turning moment due to the turning force on the elementary ring (dT)

$$dT = \frac{\tau}{R} \times 2\pi r^2 dr \times r = \frac{\tau}{R} \times 2\pi r^3 dr$$

The total turning moment is obtained by integrating the above eqn.

$$T = \frac{\pi}{16} \tau D^3$$

Torque Transmitted by Hollow circular shaft:

$$\text{The Torque } (T) = \frac{\pi}{16} \tau \left[\frac{D_o^4 - D_i^4}{D_o} \right]$$

Power transmitted by the shafts (P)

$$P = \frac{2\pi N T}{60} = \omega \times T$$

A hollow circular shaft 20mm thickness transmits 300 kW power at 200 rpm. Determine the external diameter of the shaft if the shear strain due to torsion is not exceed 0.00086. Take the modulus of rigidity = 0.8×10^5 N/mm²

Given

$T = 20$ mm; $P = 300$ kW; $N = 200$ rpm;

$C = 0.8 \times 10^5$ N/mm²; shear strain $(\phi) = 0.00086$

$D_o = D_i + 2t = D_i + (2 \times 20)$

$D_i = D_o - 40$

Using eqn $P = \frac{2\pi N T}{60}$ (a) $300000 = \frac{2\pi \times 200 \times T}{60}$

$T = 14323.9$ N-m

$T = 14323.9 \times 1000$ N-mm

$$\boxed{T = 14323900 \text{ N-mm}}$$

wn $C = \frac{\text{shear stress}}{\text{shear strain}}$ | where $C = 0.8 \times 10^5$
shear strain $= 0.00086$

by subs
shear stress $= C \times$ shear strain
$\tau = 0.8 \times 10^5 \times 0.00086$
$\tau = 68.8\ N/mm^2$

Now, using the eqn;

$$T = \frac{\pi}{16} \tau \times \left[\frac{D_o^4 - D_i^4}{D_o}\right]$$

$$= \frac{\pi}{16} \times 68.8 \left[\frac{D_o^4 - D_i^4}{D_o}\right]$$

$$1432390 = \frac{\pi}{16} \times 68.8 \left[\frac{D_o^4 - D_i^4}{D_o}\right]$$

$$\frac{1432390 \times 16 \times D_o}{\pi \times 68.8} = D_o^4 - D_i^4$$

$$1060334.6\ D_o = D_o^4 - D_i^4$$

$$1060334.6\ D_o = (D_o^2 + D_i^2)(D_o^2 - D_i^2)$$

$$1060334\ D_o = 160(D_o^2 - 40D_o + 800)(D_o - 20)$$

$$6627\ D_o = D_o^3 - 60D_o^2 + 1600D_o - 16000$$

$$D_o^3 - 60D_o^2 - 5027\ D_o - 16000 = 0 \quad \text{—} ①$$

eqn ① solved by trial & error method.

Let $D_o = 100$ mm

sub the value of D_o in LHS of the eqn

$$= 100^3 - 60(100)^2 - 5027(100) - 16000$$

$$\boxed{= -118700}$$

Let $D_o = 110$ mm

sub the value in LHS eqn

$$LHS = 110^3 - 60 \times 110^2 - 5027 \times 110 - 16000$$

when $\boxed{D_o = \text{[redacted]}36030}$

When $D_0 = 100$ mm, the LHS is negative but when $D_0 = 110$ mm, the LHS is positive. The value of D_0 is more nearer to 110 mm an 36030 len than 118710

(iii) Let $D_0 = 108$ mm; sub the value in the eqn

$$LHS = 108^3 - 60 \times 108 - 5027 \times 108 - 16000$$
$$= 1259910 - 699840 - 542916 - 16000$$
$$= 1194$$

2. A Hollow shaft is to transmit 300KW power at 80 rpm. If the shear stress is not to exceed $60\ N/mm^2$ and the internal diameter is 0.6 of the external diameter. Find the external and internal diameters assuming that the maximum torques is 1.4 times the mean.

Given

Power transmitted, $P = 300\ KW = 300 \times 10^3\ W$

speed, $N = 80$ r.p.m

Maximum shear stress, $\tau = 60\ N/mm^2$

Internal diameter, $D_i = 0.6 \times$ Ext. diameter $= 0.6 D_0$

Maximum Torque $= T_{max} = 1.4 \times T_{mean}$

Power $= P = \frac{2\pi N T}{60}$

To find (i) external diameter $[D_0]$ ii) Internal diameter $[D_i]$

soln

$$T = 60 \times P / 2\pi N = \frac{60 \times 300 \times 10^3}{2\pi \times 80} = 35809.8\ N\text{-}m$$

$$T_{max} = 1.4\,T = 1.4 \times T = 1.4 \times 35809.8\ N\text{-}m$$

$$T_{max} = 50133.7\ N\text{-}m$$

Now maximum torque transmitted by hollow shaft

$$T_{max} = \frac{\pi}{16} \times \tau \left[\frac{D_0^4 - D_i^4}{D_0}\right]$$

$$50133700 = \frac{\pi}{16} \times 60 \times \left[\frac{D_0^4 - 0.6\,D_0^4}{D_0}\right]$$

$$= \frac{\pi}{16} \times 60 \times 8.704\,D_0^3$$

$$D_0 = 170\ mm$$

where $D_i = 0.6 \times D_0$

$D_i = 102$ mm

A solid circular shaft transmits 75 kW power at 200 rpm. Calculate the shaft diameter, if the twist of shaft is not to exceed 1° to 2 metres length of shaft & shear stress is limited to 50 N/mm². Take $C = 1\times10^5$ N/mm²

<u>Given</u> Power transmitted, $P = 75\times10^3$ W, Twist $\theta = 1^\circ = \frac{\pi}{180}$ rad

speed of shaft, $N = 200$ rpm,

$\theta = 0.01745$ rad

Length of the shaft $= L = 2$ m

Maximum shear stress $= \tau = 50$ N/mm²

Modulus of rigidity $= C = 1\times10^5$ N/mm²

Power $= P = \frac{2\pi N T}{60} \Rightarrow 75\times10^3 = \frac{2\pi\times200\times T}{60}$

$T = 3580980$ N-mm

(i) Diameter of the shaft, when maximum shear stress is limited to 50 N/mm²

$T = \frac{\pi}{16}\tau D^3$; $\left[\frac{3580980\times16}{\pi\times50}\right]^{1/3} = D$

$\boxed{D = 71.3 \text{ mm}}$

(ii) Diameter of the shaft when the twist of shaft is not to exceed 1°

$\frac{T}{J} = \frac{C\theta}{L} \Rightarrow \frac{3580980}{\frac{\pi}{32}D^4} = \frac{10^5\times0.01745}{2000}$

$D = \frac{32\times2000\times3580980}{\pi\times10^5\times0.01745} = 80.4$ mm

The suitable diameter of the shaft is the greater value of two diameters given by the eqns 80.4 mm say 81 mm.

STRENGTH OF SHAFT / VARYING SECTIONS

When a shaft is made up of different lengths and the different diameters, the torque transmitted by individual sections should be calculated first. The strength of such a shaft is the minimum value of the torques

A shaft ABC of 500 mm length and length 40 mm diameter is bored, for a part of its length AB, to a 20 mm diameter and for remaining length BC to a 30 mm diameter bore. If the shear stress is not to exceed 80 N/mm², find the maximum power, the shaft can transmit at a speed of 200 rpm. If the angle of twist in the length of 20 mm diameter bore is equal to 30 mm bore diameter, find the length of the shaft

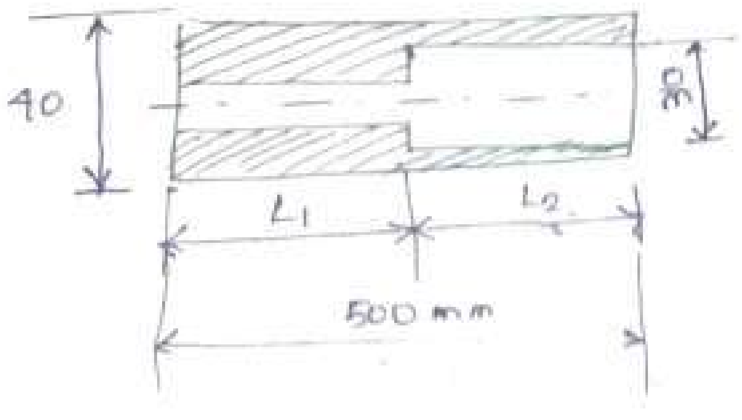
Given Total length = L = 500 mm
External dia = D = 40 mm
length of the shaft = L_1 = AB
Internal diameter of the shaft d_i = 20 mm
Length of the shaft BC = L_2
Internal dia of the shaft BC = 30 mm
Maximum shear stress, τ = 80 N/mm²
speed = N = 200 rpm
Torque transmitted by shaft AB = T_1
" " " BC = T_2

To find
1. Maximum power of the shaft
2. find the length of the shaft

solution

The torque transmitted by the hollow shaft

$$T = \frac{\pi}{16} \cdot \tau \left[\frac{D_o^4 - D_i^4}{D_o}\right]$$



Hence the torque transmitted by hollow shaft AB

$$T_1 = \frac{\pi}{16} \tau \left[\frac{D^4 - d_i^4}{D}\right]$$

$$= \frac{\pi}{16} \times 80 \left[\frac{40^4 - 20^4}{40}\right]$$

$$\boxed{T_1 = 942.5 \text{ N-mm}}$$

similarly, the torque transmitted by hollow shaft BC

$$T_2 = \frac{\pi}{16} \tau \left[\frac{D^4 - d_o^4}{D}\right] = \frac{\pi}{16} \times 80 \times \left[\frac{40^4 - 30^4}{40}\right]$$

$$\boxed{T_2 = 687.2 \text{ Nm}}$$

The safe torque (T) is transmitted by the shaft is 687.2 N-m

$$T = 687.2 \text{ N-m}$$

The power transmitted (P) $= \frac{2\pi N T}{60} = \frac{2 \times 3.14 \times 200 \times 6k}{60}$

$P = 14390$ W

Now the eqn

$$\frac{T}{J} = \frac{C \cdot \theta}{L} \quad (or) \quad \theta = \frac{TL}{CJ}$$

The safe torque and shear modulus (C) are same for given shaft

Hence angle of twist in shaft, $AB = \frac{TL}{CJ_1}$

angle of twist in shaft, $BC = TL_2 / C \cdot J_2$

Angle of twist AB = angle of twist BC

$$\frac{TL_1}{CJ_1} = \frac{TL_2}{CJ_2}$$

$$\frac{L_1}{J_1} = \frac{L_2}{J_2}$$

where J is the polar moment of inertia

$$J = \frac{\pi}{32}\left[D_o^4 - D_i^4\right]$$

$$J = \frac{\pi}{32}(40^4 - 20^4) \quad \text{—} ①$$

For shaft AB

For shaft BC

$$J = \frac{\pi}{32}\left[D_o^4 - D_i^4\right]$$

$$\therefore J = \frac{\pi}{32}\left[40^4 - 30^4\right] \quad - ②$$

① = ②

$$\frac{L_1}{\frac{\pi}{32}\left[40^4 - 20^4\right]} = \frac{L_2}{\frac{\pi}{32}\left[40^4 - 30^4\right]}$$

$$\frac{L_1}{L_2} = \frac{\left[40^4 - 20^4\right]}{\left[40^4 - 30^4\right]} = 1.37$$

(or)

$$L_1 = 1.37 L_2$$

wkt

$L_1 + L_2 = 500$

$L_2 = 500 - L_1$

sub L_2 in the eqn

$$L_1 = 1.37\left[500 - L_1\right]$$

$$= 1.37 \times 500 - 1.37 L_1$$

$$L_1 + 1.37 L_1 = 1.37 \times 500$$

$$2.37 L_1 = 1.37 \times 500$$

$$L_1 = \frac{1.37 \times 500}{2.37}$$

$$\therefore \boxed{L_1 = 289 \text{ mm}}$$

and $L_3 = 500 - 289 = 211$ mm

A steel shaft ABCD having a total length of 2.4 m consists of 3 lengths having different sections as follows. AB is hollow having outside and inside diameters of 80 mm and 50 mm respectively and BC and CD are solid. BC having a diameter of 80 mm and CD having a diameter of 70 mm. If the angle of twist is same for each section, determine the length of each section and the total angle of twist if the maximum shear stress in the hollow portion is 50 N/mm². Take $C = 8.2 \times 10^4$ N/mm²

Given Total length of the shaft, $L = 2400$ mm

Shaft AB - length $= L_1$ | Outer diameter $D_1 = 80$ mm, Inner diameter $d_1 = 50$ mm

Shaft BC - length $= L_2$ | Diameter of BC, $D_2 = 80$ mm

Shaft CD - length $= L_3$ | Diameter of CD, $D_3 = 70$ mm

Angle of twist for each section

$\theta_1 = \theta_2 = \theta_3$

Maximum shear stress in hollow portion, $\tau_1 = 50$ N/mm²

value of $C = 8.2 \times 10^4$ N/mm²

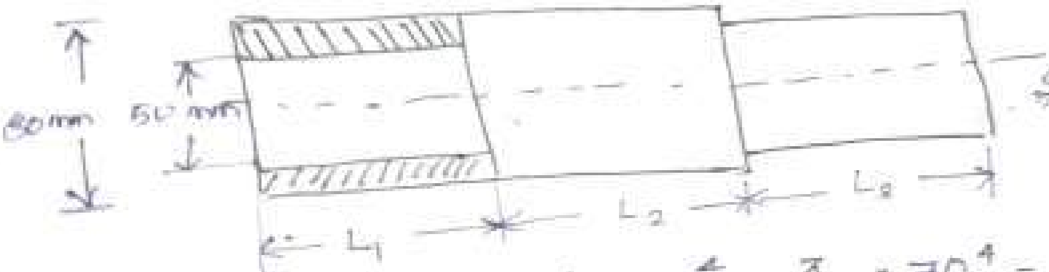
Polar M.I of each shaft is given by

For shaft AB

$$J_1 = \frac{\pi}{32}\left[D_1^4 - d_1^4\right]$$

$$J_1 = \frac{\pi}{32} \times (80^4 - 50^4)$$

Shaft BC



$= 235.8 \times 10^4$ mm⁴

For shaft CD, $J_3 = \frac{\pi}{32} D_3^4 = \frac{\pi}{32} \times 70^4$

Now using the eqn, $\frac{T}{J} = \frac{C\theta}{L}$

Hence $\theta_1 = \frac{T \cdot L_1}{J_1 \times C}$; $\theta_2 = \frac{T \cdot L_2}{J_2 \cdot C}$ & $\theta_3 = \frac{T \times L_3}{J_3 \times C}$

But $\theta_1 = \theta_2 = \theta_3$

$$\frac{TL}{C J_1} = \frac{TL_2}{C J_2} = \frac{TL_3}{C J_3}$$

$$\frac{L_1}{J_1} = \frac{L_2}{J_2} = \frac{L_3}{J_3}$$ [Torque T and C are same for each portion]

$$\frac{L_1}{340.9 \times 10^4} = \frac{L_2}{402.4 \times 10^4} = \frac{L_3}{235.8 \times 10^4}$$

$$\frac{L_1}{340.9} = \frac{L_2}{402.4} = \frac{L_3}{235.8} \quad — (1)$$

$$L_1 = \frac{340.9}{235.8} L_3 = 1.44 L_3$$

$$L_2 = \frac{402.4}{235.8} L_3 = 1.71 L_3 \quad — (2)$$

But $L_1 + L_2 + L_3 = 2.4\ m = 2400\ mm$

$$1.44 L_3 + 1.71 L_3 + L_3 = 2400$$

$$L_3 = \frac{2400}{4.15} = 578.3\ mm$$

sub the L_3 value in eqns (1) & (2)

$L_1 = 1.44 \times 578.3 = 832.75\ mm$

$L_2 = 1.71 \times 578.3 = 988.80\ mm$

As the shear stress is given in shaft AB. The angle of twist of shaft AB can be obtained by using eqn

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

For shaft AB, $$\frac{\tau_1}{\frac{D_1}{2}} = \frac{C\theta_1}{L_1}$$

$$\theta_1 = \frac{\tau_1 \times L_1}{\left(\frac{D_1}{2}\right) \times C} = \frac{50 \times 832.75}{\frac{80}{2} \times 8.2 \times 10^4}$$

$\theta_1 = 0.0126$ radian

$\theta_1 = 0.7273^\circ$

$\therefore$ Total angle of twist of whole shaft

$= \theta_1 + \theta_2 + \theta_3 = 0.727 \times 3$

$= 2.1819^\circ$

Two solid shafts AB and BC of aluminium and steel respectively are rigidly fastened together at B and attached to two rigid supports at A and C. Shaft AB is 7.5 cm in diameter and 2 m in length, shaft BC is 5.5 cm diameter and 1 m in length. A torque of 20000 N-cm is applied at the junction B. Compute the maximum shearing stress in each material. What is the angle of twist at the junction. Take modulus of Rigidity of materials $C_{Al} = 0.3 \times 10^5 N/mm^2$ and $C_{St} = 0.9 \times 10^5 N/mm^2$

Given

solid shaft AB	solid shaft BC
material = Aluminium	material = steel
length, $L_1 = 2m$	Length = $L_2 = 1m = 1000\ mm$
Dia, $d_1 = 7.5cm$	Dia $= d_2 = 5.5cm = 55\ mm$
modulus of Rigidity $C_1 = 0.3 \times 10^5 N/mm^2$	modulus of elasticity $C_2 = 0.9 \times 10^5 N/mm^2$

(Torque at junction B) = 20000 N-m

To find
1. maximum shear stress
2. Angle of twist

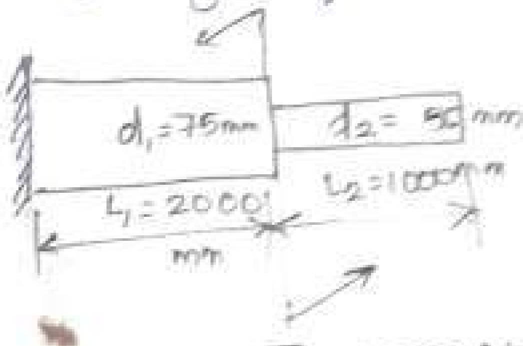
The torque is applied at junction B, hence the angle of twist in shaft AB and in shaft BC will be same ($\theta_1 = \theta_2 = \theta$)

θ_1 = Angle of twist in shaft AB

θ_2 = Angle of twist in shaft BC

T_1 = Torque transmitted to shaft AB

T_2 = Torque transmitted to shaft BC



T = 20000 N-cm

$$\frac{T}{J} = \frac{C\theta}{L}$$

$\therefore T_1 + T_2 = 20000$ N-m

For shaft AB, the above eqn becomes as

$$\frac{T_1}{J_1} = \frac{C_1 \times \theta}{L_1}$$

$$\theta_1 = \frac{T_1 L_1}{C_1 \times J_1}$$

where $J_1 = \frac{\pi}{32} d_1^4$

$= \frac{\pi}{32} \times 75^4$

$$\theta_1 = \frac{T_1 \times 2000}{\pi/32 \times 75^4 \times 0.3 \times 10^5}$$

$$\theta_1 = \frac{T_1 \times 2000 \times 32}{\pi \times 75^4 \times 0.3 \times 10^5}$$

For shaft BC, the value of θ_2 is given by

$$\theta_2 = \frac{T_2 \times L_2}{J_2 \times C_2} \quad \text{where } J_2 = \frac{\pi}{32} d_2^4$$

$$= \frac{T_2 \times 1000}{\frac{\pi}{32} \times 55^4 \times 0.9 \times 10^5} = \frac{T_2 \times 1000 \times 32}{\pi \times 55^4 \times 0.9 \times 10^5}$$

$$\theta_1 = \theta_2$$

$$\therefore \frac{T_1 \times 2000 \times 32}{\pi \times 75^4 \times 0.3 \times 10^5} = \frac{T_2 \times 1000 \times 32}{\pi \times 55^4 \times 0.9 \times 10^5}$$

$$\frac{2T_1}{75^4 \times 0.3} = \frac{T_2}{55^4 \times 0.9 \times 10^5}$$

$$T_1 = 0.576\, T_2$$

sub the value of the eqn

$$0.576\, T_2 + T_2 = 200000$$

$$1.576\, T_2 = 200000$$

$$T_2 = 126900 \text{ N-mm}$$

But $T_1 + T_2 = 200000$

$$T_1 = 200000 - T_2 = 200000 - 126900$$

$$T_1 = 73100 \text{ N-mm.}$$

From the eqn $\frac{T}{J} = \frac{\tau}{R}$

For shaft AB

$$\frac{T_1}{J_1} = \frac{\tau_1}{R_1}$$

$$\tau_1 = \frac{73100 \times 37.5}{\frac{\pi}{32} \times 75^4}$$

$$\tau_1 = 0.882 \text{ N/mm}^2$$

For shaft BC

$$\frac{T_2}{J_2} = \frac{\tau_2}{R_2}$$

$$\tau_2 = \frac{126900 \times 27.5}{\frac{\pi}{32} \times 55^4}$$

$$\tau_2 = 3.88 \text{ N/mm}^2$$

Composite shaft:

A composite shaft made up 2 (or) more different materials and behaving as a single shaft is called composite shaft. Hence composite shaft one type of shaft is rigidly solved over the another type of shaft. But the angle of twist for each shaft is equal.

Problems

1. A composite shaft consists of a steel rod 60 mm diameter surrounded by a closely fitting tube of brass. Find the outside diameter of the tube, so that when a torque of 1000 N-m is applied to the composite shaft, it will be shared equally by two materials. Take C for steel = 8.4×10^4 N/mm² and C for brass = 4.2×10^4 N/mm². Find also the maximum shear stress in each material and common angle of twist in a 4m length.

Given

Dia of the steel rod $= d = 60$ mm

Torque, $T = 1000$ N-m

$C_{steel} = 8.4 \times 10^4$ N/mm²

$C_{brass} = 4.2 \times 10^4$ N/mm²

Length of composite shaft $= L = 4$ m

Let

D = Outside diameter of the brass tube in mm

τ_s = shear stress in steel

τ_b = shear stress in brass

To find

1. outside diameter
2. Maximum shear stress
3. Common angle of twist

Solution

The inner diameter of brass tube will be equal to diameter of steel rod.

$\therefore$ Inner diameter of steel rod (or) brass tube

$d = 60$ mm

Polar M.I for brass tube is given by

$$J_b = \frac{\pi}{32}\left[D^4 - 60^4\right]$$

Let T_s = Torque transmitted by steel rod

T_b = Torque transmitted by brass tube

Polar moment of inertia for steel rod

$$J_s = \frac{\pi}{32} d^4$$

$$J_s = \frac{\pi}{32} \times 60^4 \text{ mm}^4$$

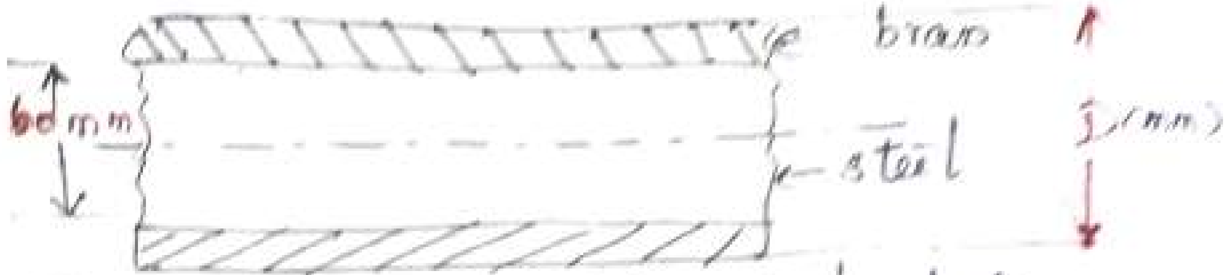
But Total torque

$$T = (T_s + T_b)$$

w.k.t $T_s = T_b$

Polar M.I for brass tube is given by

$$J_b = \frac{\pi}{32}\left[D^4 - 60^4\right] \text{mm}^4$$



T_s = Torque transmitted by steel rod &

T_b = Torque transmitted by brass tube

But, Total torque,

$$T = T_s + T_b$$

Using the eqn

$$T/J = \frac{C\theta}{L}$$

For steel rod,

$$T_s = \frac{C_s \times \theta_s \times J_s}{L_s}$$

$$T_b = \frac{C_b \times \theta_b \times J_b}{L_b}$$

But $L_s = L_b = L$

Hence the above eqn becomes

$$C_s \times J_s = C_b \times J_b$$

(or)

$$8.4 \times 10^4 \times \frac{\pi}{32} \times 60^4 = 4.2 \times 10^4 \times \frac{\pi}{32}\left[D^4 - 60^4\right]$$

$$\frac{8.4 \times 60^4}{4.2} = (D^4 - 60^4)$$

$$3 \times 60^4 = D^4 \quad \text{(or)} \quad D = 78.98 \text{ mm}$$

(i) shear stresses in each material

Using the eqn

$$\frac{T}{J} = \frac{\tau}{R}$$

For a steel rod, we have

$$\tau_s = T_s \times \frac{d}{2} \Big/ J_s = \frac{500 \times 10^3 \times \frac{60}{2}}{\frac{\pi}{32} \times 60^4} = 11.79\ N/mm^2$$

For a brass tube, we have

$$\tau_b = \frac{T_b \times \frac{D}{2}}{J_b} = \frac{500 \times 10^3 \times \frac{79}{2}}{\frac{\pi}{32}[D^4 - 60^4]} = \frac{500 \times 10^3 \times \frac{79}{2}}{\frac{\pi}{32}[79^4 - 60^4]}$$

$$\boxed{\tau_b = 7.76\ N/mm^2}$$

common angle of twist

$$T_s = \frac{C_s \times \theta_s \times J_s}{L_s} \; ; \quad \theta_s = \frac{T_s \times L_s}{C_s \times J_s}$$

$$\theta_s = \frac{500 \times 10^3 \times 4000}{8.4 \times 10^4 \times \frac{\pi}{32} \times 60^4} = 0.01871 \text{ radians}$$

hence $\theta_s = 0.01871 \times \frac{180}{\pi}$ degrees $= 1.072°$

But the angle of twist will be equal. The common angle of twist will be equal to the angle of twist in any shaft

Using eq(1), we got

$$T_s = \frac{C_s \times \theta_s \times J_s}{L_s} \; ; \quad \theta_s = \frac{T_s \times L_s}{C_s \times J_s}$$

$$\theta_s = \frac{500 \times 10^3 \times 4000}{8.4 \times 10^4 \times \frac{\pi}{32} \times 60^4} = 0.01871 \text{ radians}$$

$\theta_s = 0.01571 \times \frac{180}{\pi}$ degrees $= 1.072°$

But angle of twist in each shaft will be equal.

Combined Bending and Torsion:

When a shaft is transmitting torque and it is subjected to shear stresses Due to the BM, bending stresses are also set up in the shaft Hence each particle of the shaft is subjected to shear stresses and bending stress

The principal stress and maximum shear stress, when a shaft is subjected to bending & torsion.

Consider any point on the cross-section of the shaft

T - Torque ; D - Diameter ; M - Bending moment

The Torque (T) will produce shear stress at the point, where BM will produce bending stress.

Let q - shear stress at the point produced by T

σ - Bending stress at the point produced by B.M.

The shear stresses at any point due to Torque (T)

$$\frac{q}{r} = \frac{T}{J} \; ; \qquad q = \frac{T}{J} \times r$$

The bending stress at any point due to the Torque (T)

$$\frac{M}{I} = \frac{\sigma}{y} \quad (\text{or}) \quad \sigma = \frac{M \times y}{I}$$

Angle θ made by the plane of maximum shear with normal cross section

$$\tan 2\theta = 2\tau/\sigma$$

The bending stress and shear stress is maximum at this point on the shaft surface $r = R = \frac{D}{2}$; $y = \frac{D}{2}$

Let $\sigma_b = \frac{M}{I} \times \frac{D}{2} = \frac{M}{\frac{\pi}{64} D^4} \times \frac{D}{2} = \frac{32 M}{\pi D^3}$

τ = Maximum shear stress on the shaft surface.

$$\tau = \frac{T}{J} \times R = \frac{T}{\frac{\pi}{32} D^4} \times \frac{D}{2} = \frac{16 T}{\pi D^3}$$

$$\tan 2\theta = \frac{2 \times \frac{16 T}{\pi D^3}}{\frac{32 M}{\pi D^3}} = \frac{T}{M}$$

Major principal stress $= \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$

$$= \frac{32 M}{2 \times \pi D^3} + \sqrt{\left(\frac{32 M}{2 \times \pi D^3}\right)^2 + \left(\frac{16 T}{\pi D^3}\right)^2}$$

$$= \frac{16}{\pi D^3}\left(M + \sqrt{M^2 + T^2}\right)$$

Minor principal stress $= \frac{16}{\pi D^3}\left(M - \sqrt{M^2 + T^2}\right)$

Maximum shear stress = $\frac{\text{Major principal stress} - \text{Minor principal stress}}{2}$

$= \frac{16}{\pi D^3}\left(\sqrt{M^2+T^2}\right)$

For a Hollow shaft:

Major principal stress $= \frac{16 D_0}{\pi(D_0^4 - D_1^4)}\left(M + \sqrt{M^2+T^2}\right)$

Minor principal stress $= \frac{16 D_0}{\pi(D_0^4 - D_1^4)}\left(M - \sqrt{M^2+T^2}\right)$

Major shear stress $= \frac{16 D_0}{\pi(D_0^4 - D_1^4)}\left(\sqrt{M^2+T^2}\right)$

Problems:

A solid shaft of diameter 80 mm is subjected to the twisting moment of 8 MN-mm and a bending moment of 5 MN-mm at a point. Determine (i) principal stresses (ii) position of the point, which they act.

Given

Diameter of the shaft, D = 80 mm

Twisting moment, T = 8 MN-mm

Bending moment, M = 5 MN-mm

Soln

The major principal stress $= \frac{16}{\pi D^3}\left(M + \sqrt{M^2+T^2}\right)$

$= 16\left[5\times10^6 + \sqrt{(5\times10^6)^2 + (8\times10^6)^2}\right] = \frac{16\times10^6}{\pi\times80^3}\left(5+\sqrt{25+64}\right)$

$= 143.57\ N/mm^2$

Minor principal stress $= \frac{16}{\pi D^3}\left[M - \sqrt{M^2+T^2}\right)$

$= \frac{16}{(3.14\times(0.08)^3}\left[5\times10^6 - \sqrt{(5\times10^6)^2 + (8\times10^6)^2}\right)$

$= 44.1\ N/mm^2$

Position of the plane = $\tan 2\theta = T/M = \frac{8\times10^6}{5\times10^6} = 1.6$

$2\theta = \tan^{-1}(1.6)$

$\therefore 2\theta = \frac{\tan^{-1}(1.6)}{\cancel{M}}$

$\theta = 28°59'$

Expression of strain energy in a body due to its Torsion

$R = \frac{D}{2}$; τ - shear stress; l - length of shaft;

$R = \frac{D}{2}$; | τ - shear stress on the surface of shaft

C - shear modulus or modulus of rigidity | U - Total strain energy

where $\vec{J}$ = Polar M.I of the shaft

$= \frac{\pi}{32} D^4$ for solid shaft

Total strain energy of "solid shaft" due to the torsion

$$U = \frac{\tau^2}{4C} \times V \quad \left[\therefore \text{volume, } V = \pi R^2 l\right]$$

Total strain energy of "hollow shaft"

$$U = \frac{\tau^2}{4CD^2}(D^2 + d^2) \times V$$

where $V = \frac{\pi}{4}(D^2 - d^2) \cdot l$

Problems

1. Determine the maximum strain energy stored in a solid shaft of diameter 10 cm and length 1.25 m, if the maximum allowable shear stress is 50 N/mm². Take $C = 8 \times 10^4$ N/mm²

Given

Diameter of the shaft, D = 10 cm

$\therefore$ Area of the shaft $A = \frac{\pi}{4} \times D^2 = 7854$ mm²

length of the shaft, $l = 1.25$ m

volume of the shaft, $V = A \times l = 78.54 \times 125$

$V = 9817.5 \times 10^3$ mm³

Maximum allowable shear stress

$\tau = 50$ N/mm² (shear stress is maximum on the surface of the shaft)

Modulus of rigidity, $C = 8 \times 10^4$ N/mm²

Let U = shear strain energy in the shaft

$$\therefore U = \frac{\tau^2}{4C} \times V = \frac{50^2}{4 \times 8 \times 10^4} \times 9817.5 \times 10^3$$

$\boxed{U = 76699\ N\text{-}mm}$

The external and internal diameters of a hollow shaft are 40 cm and 20 cm. Determine the maximum strain energy stored in a hollow shaft, if the maximum allowable shear stress is 50 N/mm² and the length of the shaft is 5m. Take $C = 8 \times 10^4$ N/mm²

Given: Internal dia (d) = 20 cm = 200mm

external dia (D) = 40cm = 400 mm

$\therefore$ Area of the cross section, $A = \frac{\pi}{4}[40^2 - 20^2]$

$A = 942.4\ mm^2$

Maximum allowable shear stress (τ) = 50 N/mm²

Length of the shaft (l) = 5m

$\therefore$ volume of the shaft (V) = $A \times l = 942.4 \times 500$

$V = 471235 \times 10^3\ mm^3$

Modulus of rigidity, $C = 8 \times 10^4$ N/mm² | where U = strain energy stored

$$U = \frac{\tau^2}{4CD^2}(D^2 + d^2) \times V$$

$$= \frac{50^2}{4 \times 8 \times 10^4 \times 400^2}[400^2 + 200^2] \times 471235 \times 10^3$$

$\therefore\ U = 4601.9$ N-m

A solid circular shaft of 10cm diameter of length 4m is transmitting 112.5 KW power at 150 rpm. Determine (i) the maximum shear stress induced in the shaft & (ii) strain energy stored in the shaft. Take $C = 8 \times 10^4$ N/mm².

Given Diameter of the shaft, D = 10 cm

length of the shaft, l = 4m

Power of the shaft} = P = 112.5 KW

shaft speed = N = 150 rpm

Modulus of elasticity = C = 8×10^4 N/mm²

To find

i) shear stress τ. ii) strain energy (U)

Power of the shaft $= P = \frac{2\pi N T}{60}$; $112.5 \times 10^{3} = \frac{2\pi \times 150 \times T}{60}$

$$T = \frac{112.5 \times 10^{3} \times 60}{2\pi \times 150} = 7159000 \text{ N-mm}$$

we know, $T^{*} = \frac{\pi}{16} \times \tau \times D^{3} = \frac{7159000 \times 16}{\pi \times 10^{6}}$

$$\tau^{*} = 36.5 \text{ N/mm}^2$$

Using strain energy

$$U = \frac{\tau^2}{4C} \times \text{Volume of the shaft}$$

$$= \frac{36.5^2}{4 \times 8 \times 10^4} \times \text{Volume of the shaft}$$

$$V = \frac{\pi}{4} D^2 \times l = \frac{\pi}{4} \times 100^2 \times 4000$$

$$\therefore \quad \boxed{U = 130793 \text{ N-mm}}$$

The shaft is subjected to constant torque (T)

$$\frac{T}{J} = \frac{C\theta}{L} \quad \Big| \quad J = \frac{\pi}{32} D_2^4$$

① Determine the angle of twist and maximum shear stress developed in a shaft which tapers uniformly from a diameter of 160mm to a diameter of 240mm. The length of the shaft is 2m and transmits a torque of 48 kN-m. Take the value of modulus of rigidity for shaft material as 80GN/m²

Given

$D_1 = 160\text{mm} = 0.16\text{m}$

$D_2 = 240\text{mm} = 0.24\text{m}$

$L = 2\text{m}$

$T = 48 \text{ kN-m}$

$C = 80 \times 10^{9} \text{ N/m}^2$

To find

① maximum shear stress developed

② Angle of twist, (θ)

Solution

$$\theta = \frac{32 T}{\pi C} \times \frac{1}{3K} \left[\frac{1}{D_1^3} - \frac{1}{D_2^3} \right]$$

$K = \frac{D_2 - D_1}{L} = \frac{0.240 - 0.160}{2} = 0.04$

$$\theta = \frac{32 \times 48 \times 10^3}{\pi \times 80 \times 10^9} \times \frac{1}{3 \times 0.04}\left[\frac{1}{0.16^3} - \frac{1}{0.24^3}\right]$$

$\theta = 0.00875$ radians $= 0.00875 \times \frac{180}{\pi} = 0.501$

ii) Maximum shear stress developed:

wkt $T = \frac{\pi}{16} \times \tau \times D^3$ | For a given torque, the shear stress on the surface of the shaft will be maximum, where diameter is minimum. Hence, the smaller diameter, the shear stress will be maximum.

$T = \tau_{max} \times \frac{\pi}{16} \times D_1^3$ where D_1 = smaller diameter $= 0.16$ m

$48 \times 10^3 = \tau_{max} \times \frac{\pi}{16} \times 0.16^3$ | $\boxed{\tau_{max} = 59.68\ MN/m^2}$

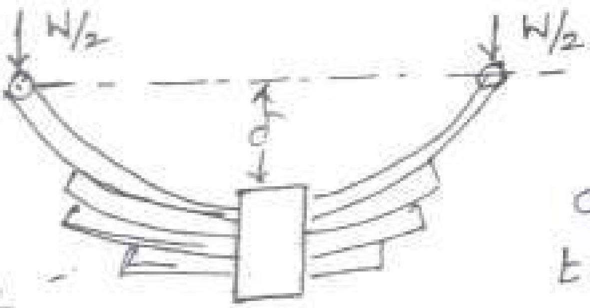
SPRINGS

Springs are the elastic bodies which absorbs energy due to resilience. The absorbed energy may be released as and when required. A Spring which is capable of absorbing the greatest amount of energy for the given stress, without getting permanently distorted. There are 2 types of springs are

1. Laminated (or) leaf springs   2. Helical springs

Laminated (or) Leaf spring:-

The laminated springs are used to absorb the shocks in railway wagons, coaches and road vehicles.



b = width of each plate
n = No. of plates
l = span of spring
σ = Maximum bending stress
t = Thickness of each plate
W = point load
δ = original deflection

B.M at the center = load at one end $\times \frac{l}{2}$

$$M = \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$$

The M.I of each plate

$$I = \frac{bt^3}{12}$$

Total resisting moment by 'n' plates

$= n \times M = n \times \frac{\sigma \cdot bt^2}{6}$

sub the value (σ) & 'R' in eqn

$$\delta = \frac{\sigma \cdot l^2}{4Et}$$

Problems

1. A Leaf spring carries a central load of 3000 N. The leaf is made up of 10 steel plates 5 cm wide and 6 mm thick. If the bending stress is limited to 150 N/mm². Determine 1. length of the spring 2. Deflection at the centre of spring.

Given

central load, $W = 3000$ N

No. of plates, $n = 10$

width of each plate, $b = 5$ cm

Thickness, $t = 6$ mm

Bending stress, $\sigma = 150$ N/mm²

Modulus of Elasticity, $E = 2 \times 10^5$ N/mm²

∴ l - length of spring ; δ = deflection at the spring center

where $\sigma = \frac{3Wl}{2nbt^2}$; $150 = \frac{3 \times 3000 \times l}{2 \times 10 \times 50 \times 6^2}$

where, $\boxed{l = 600\text{mm}}$

For deflection; $\delta = \frac{\sigma l^2}{4Et} = \frac{150 \times 600^2}{4 \times 2 \times 10^5 \times 6} = 11.25$ mm

2. A laminated spring 1 m long is made up of plates each 5 cm wide and 1 cm thick. If the bending stress in a plate is limited to 100 N/mm² how many plates would be required to enable the spring to carry a central point load of 2 KN? If $E = 2.1 \times 10^5$ N/mm² What is the deflection under the load.

Given length of spring, $l = 1 m = 1000 mm$
width of each plate, $b = 5 cm = 50 mm$
Thickness of each plate, $t = 1 cm = 10 mm$
Bending stress, $\sigma = 100 N/mm^2$
central load on spring, $W = 2 kN$
Young's modulus, $E = 2.1 \times 10^5 N/mm^2$

Let n = No. of plates; δ = deflection under the load.

Using the eqn, $\sigma = \frac{3Wl}{2nbt^2} = \frac{3 \times 2000 \times 1000}{2 \times n \times 50 \times 10^2}$

$n = \frac{3 \times 2000 \times 1000}{100 \times 2 \times 50 \times 100} = 6$

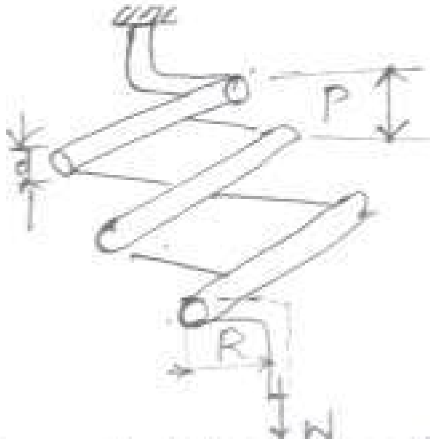
Deflection under load: deflection $(\delta) = \frac{\sigma \times l^2}{4E \times t}$

$\sigma = 100 N/mm^2$; $l = 1000 mm$;
$E = 2.1 \times 10^5 N/mm^2$; $t = 10 mm$

$\delta = \frac{100 \times 1000^2}{4 \times 2.1 \times 10^5 \times 10} = 11.9 mm$

Helical springs - Helical springs are the thick spring wires which coiled into helix. They are two types
1. close-coiled helical springs 2. Open coiled helical springs

close coiled helical springs:
These are the springs in which helix angle is very small (or) in other words the pitch b/w two adjacent turns is small.



"Expression for max. shear stress induced in wire"
The figure shows close-coiled spring (helical) subjected to an axial load.
d - diameter of spring wire
P - pitch of helical spring
n - Number of coils
R - mean Radius of spring coil
W - Axial load on spring
C - Modulus of rigidity
τ - Max shear stress
θ - Angle of twist
δ - deflection of spring due to axial load
l - length of wire

Now, twisting moment on wire
$T = W \times R$
But twisting moment is also given by
$T = \frac{\pi}{16} \tau d^3$

Equating the eqns ① & ② — The eqn given the max shear stress induced in the wire

$$W \cdot R = \frac{\pi}{16} \tau d^3$$

$$\boxed{\tau = \frac{16 W \cdot R}{\pi d^3}}$$

Expression for deflection of spring:

Now length of one coil $= \pi D$ (or) $2\pi R \times n$

∴ As the every section of wire is subjected to torsion hence the strain energy stored by the spring

$$U = \frac{\tau^2}{4C} \times \text{volume} = \frac{\tau^2}{4C} \times \text{vol}$$

$$= \frac{16 W \cdot R^2}{\pi d^3} \quad \& \text{ volume} = \frac{\pi}{4} d^2 \times \text{Total length of the wire}$$

$$= \frac{32 W^2 R^3 \cdot n}{C d^4}$$

WKT (workdone on the spring) = (Average load) × Deflection

∴ Deflection $= \delta$

$$= \frac{1}{2} W \times \delta$$

we get Equating the workdone on spring to the energy stored

$$\frac{1}{2} W \cdot \delta = \frac{32 W^2 R^3 \cdot n}{C d^4}; \quad \delta = \frac{64 W R^3 \cdot n}{C d^4}$$

expression of the spring stiffness

$$s = \text{load/unit deflection} = \frac{W}{\delta} = \frac{W}{\frac{64 W R^3 \cdot n}{C d^4}}$$

$$s = \frac{C d^4}{64 \cdot R^3 \cdot n}$$

<u>Note</u> - The solid length of spring means the distance b/w the coils, when the coils are touching each other there is no gap b/w the coils. The solid length is given by

solid length = No. of coils × dia of the wire

Problems

A closely coiled helical spring is to carry a load of 500N. Its mean coil diameter is to 10 times that the wire diameter. calculate these diameters if the maximum shear stress in the material of spring is to be 80 N/mm²

load on the spring, $W = 500\,N$
max shear stress, $\tau = 80\ N/mm^2$
diameter of wire, d; D = mean diameter of coil
$D = 10d$
Using the eqn.

$$\tau = \frac{16WR}{\pi d^3} \Rightarrow 80 = \frac{16 \times 500 \times \frac{D}{2}}{\pi d^3}$$

$$80 \times \pi d^3 = 8000 \times 5d$$

$$d^2 = \frac{8000 \times 5}{80 \times \pi} = 159.25$$

$$d = 12.6\ mm$$

$D = 10 \times d$;
$= 10 \times 1.26 = 12.6\,cm$

A closely coiled helical spring of round wire 10 mm in diameter having 10 complete turns with a mean diameter of 12 cm is subjected to axial load of 200 N. Determine (i) the deflection of spring (ii) maximum shear stress in the wire (iii) stiffness of spring. Take $C = 8 \times 10^4\ N/mm^2$

Given: Dia of wire $= d = 10\,mm$; No. of turns $(n) = 10$
Mean dia of coil $= D = 12\,cm = 120\,mm$
Radius of coil: $R = \frac{D}{2} = 60\,mm$; Axial load, $W = 200\,N$
Modulus of rigidity, $C = 8 \times 10^4\ N/mm^2$
deflection of spring $= \delta$; τ - Maximum shear stress in wire
s = stiffness of spring

WKT

$$\delta = \frac{64\,WR^3 n}{cd^4} = \frac{64 \times 200 \times 60^3 \times 10}{8 \times 10^4 \times 10^4}$$

$$\delta = 34.5\ mm$$

$$\tau = \frac{16WR}{\pi d^3} = \frac{16 \times 200 \times 60}{\pi \times d^3} = \frac{16 \times 200 \times 60}{\pi \times 10^3} = 61.1$$

stiffness of the spring $(s) = \frac{W}{\delta} = \frac{200}{34.5} = 5.$

The stiffness of a closed coil helical spring is 1.5 N/mm of compression under a maximum load of 60N. The maximum shearing stress produced in a wire of spring is 125 N/mm². The solid length of the spring (when the coils are touching) is given as 5 cm. Find (i) diameter of wire

Given

stiffness of spring, $s = 1.5\ N/mm$

Load on the spring, $W = 60\ N$

Maximum shear stress, $\tau = 125\ N/mm^2$

solid length of the string $= l = 5\ cm$

Modulus of rigidity, $C = 4.5 \times 10^4\ N/mm^2$

d = diameter of wire; D = mean diameter of coil

n = No. of coils $\quad R$ = mean radius $= \frac{D}{2}$

To find:

$$s = \frac{Cd^4}{64 \cdot R^3 \cdot n} \Rightarrow 1.5 = \frac{4.5 \times 10^4 \times d^4}{64 \times R^3 \times n}$$

$$d^4 = \frac{1.5 \times 64 \times R^3 \times n}{4.5 \times 10^4} = 0.002133\ R^3 n$$

Using the eqn $\quad \tau = 16W \times R / \pi d^3; \quad R = \frac{125 \times \pi d^3}{16 \times 60}$

$$\boxed{R = 0.409 d^3}$$

sub the value of 'R' in eqn

$$d^4 = 0.002133 \times (0.40906 d^3)^3 \times n$$

$$= 0.002133 \times 0.4096^3 \times d^9 \times n$$

$$\frac{d^9 \cdot n}{d} = \frac{1}{0.00014599}; \quad d^5 \cdot n = \frac{1}{0.00014599}$$

solid length $(l) = n \times d$ (or) $50 = n \times d$;

$n = 50/d$

sub the value of 'n' eqn

$$d^5 \times \frac{50}{d} = \frac{1}{0.00014599}$$

$$d^4 = (136.9)^1 = 3.42\ mm$$

sub the value in eqn

$$n = \frac{50}{d} = \frac{50}{3.42} = 14.6 \simeq 15$$

Also from the eqn

$$R = 0.40906 \times d^3 = 0.409 \times 3.42^3$$

$$= 16.36\ mm$$

Mean dia of the coil $= D = 2R = 32.72\ mm$

Formulas

1. The relation of maximum shear stress induced in the shaft subjected to turning moment is given by

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

τ - max shear stress
R - Radius of shaft
C - modulus of rigidity

2. When a circular shaft is subjected to torsion the shear stress at any point varies linearly from the axis to the surface

$$\frac{\tau}{R} = \frac{q}{r}$$

3. The shear stress is maximum on the shaft surface and zero to shaft axis

4. The torque transmitted by a solid shaft is given by

$$T = \frac{\pi}{16} \tau \cdot D^3$$

5. The torque transmitted by a hollow circular shaft is given by

$$T = \frac{\pi}{16} \tau \left[\frac{D_o^4 - D_i^4}{D_o}\right]$$

D_o - outside diameter
D_i - internal diameter

6. Relation b/w polar M.I and shear stress is given as

$$\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}$$

J - Polar M.I
$= \frac{\pi}{32} D^4$ (solid)
$= \frac{\pi}{32} [D_o^4 - D_i^4]$ (hollow)

7. Ratio of Polar M.I to the radius of the shaft is known as polar modulus (or) torsional section modulus. It is denoted by (Z_p)

$$\therefore Z_p = \frac{J}{R}$$

8. The product of modulus of rigidity and polar M.I is known as torsional rigidity (or) stiffness

$$\text{Torsional rigidity} = C \times J = T \cdot L / \theta$$

where L = 1 metre, θ = 1 radian

9. The strain energy stored in a shaft due to torsion

$$U = \frac{\tau^2}{4C} \cdot V \quad \text{(for solid)}$$

$$U = \frac{\tau}{4C \cdot D^2} [D^2 + d^2] \cdot V \quad \text{(for Hollow)}$$

10. Springs are the elastic bodies which absorbs energy due to resilience. There are 2 important types of springs.

1. Laminated (or) Leaf springs 2. Helical springs

The maximum stress developed in the plates of leaf spring

$$\sigma = \frac{3Wl}{2nbt^2}$$

W - point load of leaf spring
l - Length of the leaf spring
n - No. of the plates
b - width of each plate
t - Thickness of each plate

The central deflection (δ) of the laminated spring is given by

$$\delta = \frac{\sigma l^2}{4Et}$$

Helical springs are those springs coiled into a helix. They are 2 types

1 closed coil helical spring 2 open coiled helical spring

The maximum shear stress is induced in the wire of closed-coil helical spring which carries an axial load

$$\tau = \frac{16WR}{\pi d^3}$$

W - axial spring load
R - mean radius of spring coil
d - diameter of spring wire

For a close coiled helical spring, which carries an axial load, we have

(i) strain energy stored

$$U = \frac{32W^2R^3n}{Cd^4}$$

(ii) The deflection of the spring at center due to axial load

$$\delta = \frac{64WR^3n}{Cd^4}$$

(iii) The stiffness of the spring

$$s = \frac{Cd^4}{64R^3n}$$

n = No. of coils