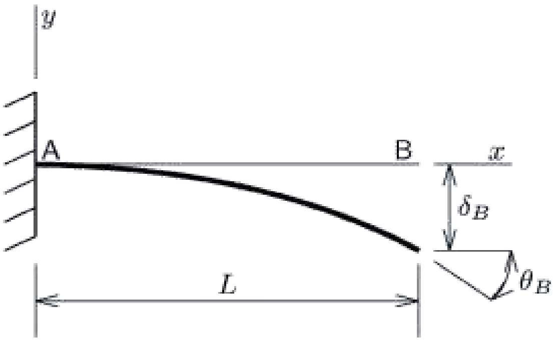
INTRODUCTION

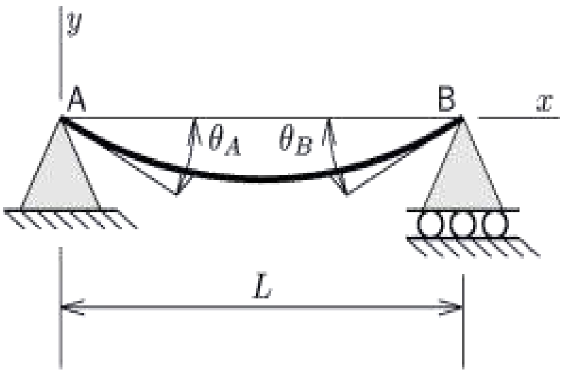
SLOPE OF A BEAM:

- ✓ slope at any section in a deflected beam is defined as the angle in radians which the tangent at the section makes with the original axis of the beam.
- ✓ slope of that deflection is the angle between the initial position and the deflected position.

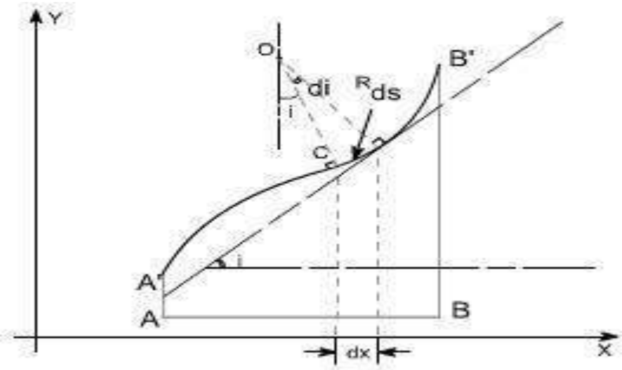
DEFLECTION OF A BEAM:

- ✓ The deflection at any point on the axis of the beam is the distance between its position before and after loading.
- ✓ When a structural is loaded may it be Beam or Slab, due the effect of loads acting upon it bends from its initial position that is before the load was applied. It means the beam is deflected from its original position it is called as Deflection.





BASIC DIFFERENTIAL EQUATION:



Consider a beam AB which is initially straight and horizontal when unloaded. If under the actionof loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holdsgood.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x andy, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Further, let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di.

But for the deflected shape of the beam the slope i at any point C is defined,

$\tan i = \frac{dy}{dx}$ ......(1) or $i = \frac{dy}{dx}$ Assuming $\tan i = i$

Futher

$ds = R\,di$

however,

$ds = dx$ [usually for small curvature]

Hence

$ds = dx = R\,di$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R}$ or $\frac{d^2y}{dx^2} = \frac{1}{R}$

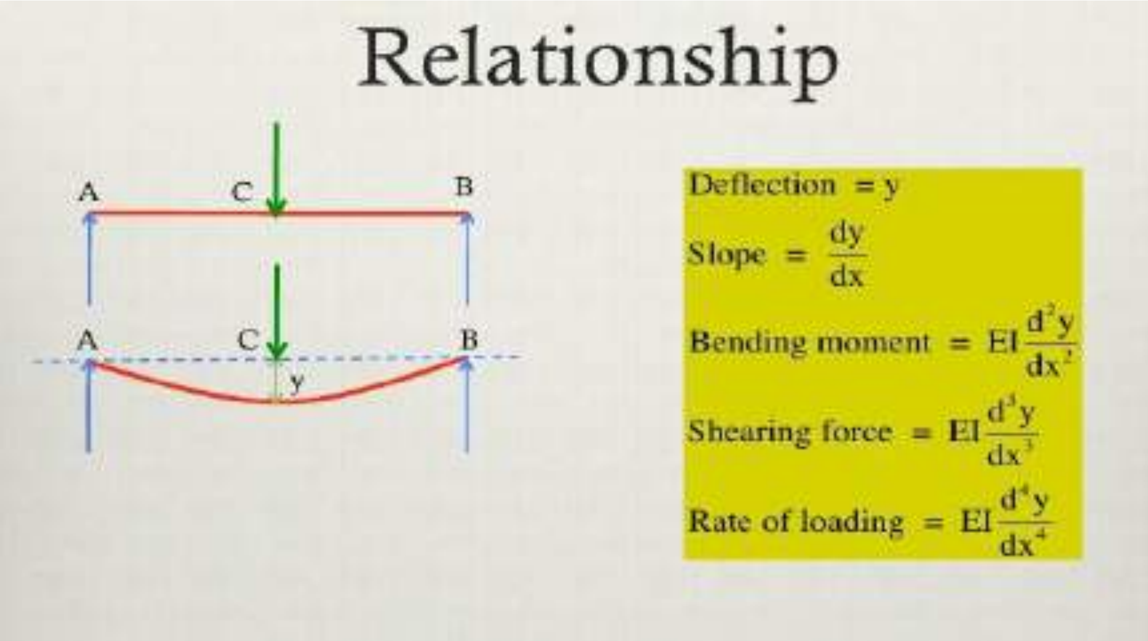
From the simple bending theory

$\frac{M}{I} = \frac{E}{R}$ or $M = \frac{EI}{R}$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry.



METHODS FOR FINDING THE SLOPE AND DEFLECTION OF BEAMS:

- Double integration method
- Moment area method
- Macaulay's method
- Conjugate beam method
- Strain energy method

DOUBLE INTEGRATION METHOD:

- ✓ The double integration method is a powerful tool in solving deflection and slope of a beam at any point because we will be able to get the equation of the elastic curve.
- ✓ This method entails obtaining the deflection of a beam by integrating the differential equation of the elastic curve of a beam twice and using boundary conditions to determine the constants of integration.
- ✓ The first integration yields the slope, and the second integration gives the deflection.

CONJUGATE BEAM:

- ✓ Conjugate beam is defined as the imaginary beam with the same dimensions (length) as that of the original beam but load at any point on the conjugate beam is equal to the bending moment atthat point divided by EI.
- ✓ Slope on real beam = Shear on conjugate beam
- ✓ Deflection on real beam = Moment on conjugate beam

PROPERTIES OF CONJUGATE BEAM:

- ✓ The length of a conjugate beam is always equal to the length of the actual beam.
- ✓ The load on the conjugate beam is the M/EI diagram of the loads on the actual beam.
- ✓ A simple support for the real beam remains simple support for the conjugate beam.
- ✓ A fixed end for the real beam becomes free end for the conjugate beam.
- ✓ The point of zero shear for the conjugate beam corresponds to a point of zero slope for the real beam.
- ✓ The point of maximum moment for the conjugate beam corresponds to a point of maximum deflection for the real beam.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

Fig. 12.3

Now $R_A = R_B = \frac{W}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2 y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2 y}{dx^2} = \frac{W}{2} \times x \quad ...(i)$$

On integration, we get

$$EI \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \quad ...(ii)$$

where C_1 is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at $x = \frac{L}{2}$, slope $\left(\frac{dy}{dx}\right) = 0$ (As the maximum deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (ii), we get

$$0 = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or

$$C_1 = -\frac{WL^2}{16}$$

Substituting the value of C_1 in equation (ii), we get

$$EI \frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \quad ...(iii)$$

The above equation is known the *slope equation*. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A, $x = 0$ and hence slope at A will be obtained by substituting $x = 0$ in equation (iii).

$$\therefore \quad EI\left(\frac{dy}{dx}\right)_{at\ A} = \frac{W}{4}\times 0 - \frac{WL^2}{16}$$

$\left[\left(\frac{dy}{dx}\right)_{at\ A}\right.$ is the slope at A and is represented by $\left.\theta_A\right]$

$$EI \times \theta_A = -\frac{WL^2}{16}$$

or

$$\therefore \quad \theta_A = -\frac{WL^2}{16EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \quad \theta_B = \theta_A = -\frac{WL^2}{16EI} \quad \ldots(12.6)$$

Equation (12.6) gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation (*iii*). Hence integrating equation (*iii*), we get

$$EI \times y = \frac{W}{4}\cdot\frac{x^3}{3} - \frac{WL^2}{16}x + C_2 \quad \ldots(iv)$$

where C_2 is another constant of integration. At A, $x = 0$ and the deflection (y) is zero.

Hence substituting these values in equation (*iv*), we get

$$EI \times 0 = 0 - 0 + C_2$$

or

$$C_2 = 0$$

Substituting the value of C_2 in equation (*iv*), we get

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2 . x}{16} \quad \ldots(v)$$

The above equation is known as *the deflection equation*. We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where $x = \frac{L}{2}$. Let y_c represents the deflection at C. Then substituting $x = \frac{L}{2}$ and $y = y_c$ in equation (*v*), we get

$$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\times\left(\frac{L}{2}\right)$$

$$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$$

$$= -\frac{2WL^3}{96} = -\frac{WL^3}{48}$$

$$\therefore \quad y_c = -\frac{WL^3}{48EI}$$

(Negative sign shows that deflection is downwards)

$$\therefore \quad \text{Downward deflection, } y_c = \frac{WL^3}{48EI} \quad \ldots(12.7)$$

PROBLEMS:

1.A beam 6 m long, simply supported at its ends, is carrying a point load of 50 KN at its centre. The moment of inertia of the beam is 78 x 10^6 mm^4. If E for the material of the beam = 2.1 X 10^5 N/mm^2. calculate deflection at the centre of the beam and slope at the supports.

GIVEN DATA:

L = 6 m

W = 50 KN = 50 X 10^3 N

I = 78 X 10 mm^4

E = 2.1 X 10^5 N/mm^2

SOLUTION:

1. DEFLECTION AT THE CENTRE OF THE BEAM,

$y_c = WL^3 / 48\ EI$

$= 50000 \times 6000^3 / (48 \times 2.1 \times 10^5 \times 78 \times 10^6)$

= 13.736 mm.

2. SLOPE AT THE SUPPORTS,

$\Theta_A = \Theta_B = - WL^2 / 16 EI$

$= 50000 X 6000^2 / (16 X 2.1 X 10^5 X 78 X 10^6)$

= 0.06868 radians.

2. A beam carries 4 m long simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1° , find the deflection at the centre of the beam.

GIVEN DATA:

L = 4 m

$\Theta_A = \Theta_B = 1° = 1° X (\pi /180) = 0.01745$ radians.

SOLUTION:

1. DEFLECTION AT THE CENTRE OF THE BEAM,

$\Theta_A = \Theta_B = - WL^2 / 16 EI$

$0.01745 = WL^2 / 16 EI$

$y_c = WL^3 / 48 EI$

$= WL^2 / 16 EI X (L/3)$

$= 0.01745 X (4000/3)$

= 23.26 mm.

3. A beam 3 m long, simply supported at its ends, is carrying a point load W at the centre. If the slope at the ends of the beam should not exceed 1° , find the deflection at the centre of the beam.

GIVEN DATA:

L = 3 m

$\Theta_A = \Theta_B = 1° = 1 X (\pi /180) = 0.01745$ radians.

SOLUTION:

1.DEFLECTION AT THE CENTRE OF THE BEAM,

$\Theta_A = \Theta_B = - WL^2 / 16 EI$

$0.01745 = WL^2 / 16 EI$

$y_c = WL^3 / 48 EI$

$= WL^2 / 16 EI X (L/3)$

$= 0.01745 X (3000/3)$

= 17.45 mm.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED WITH A UNIFORMLY DISTRIBUTED LOAD:

- ✓ A simply supported beam AB of length L and carrying a uniformly distributed load of w per unit length over the entire length is shown in fig.
- ✓ The reactions at A and B will be equal.
- ✓ Also, the maximum deflection will be at the centre of the beam.
- ✓ Each vertical reaction = (w X L)/2

Fig 12.6

$$\therefore \quad R_A = R_B = \frac{w \times L}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x - w \times x \times \frac{x}{2} = \frac{w \cdot L}{2} \cdot x - \frac{w \cdot x^2}{2}$$

But B.M. at any section is also given by equation (12.3), as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = \frac{w \cdot L}{2} x - \frac{w \cdot x^2}{2}$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{w \cdot L}{2} \cdot \frac{x^2}{2} - \frac{w}{2} \cdot \frac{x^3}{3} + C_1 \qquad \ldots(i)$$

where C_1 is a constant of integration.

Integrating the above equation again, we get

$$EI.y = \frac{w \cdot L}{4} \cdot \frac{x^3}{3} - \frac{w}{6} \cdot \frac{x^4}{4} + C_1x + C_2 \qquad \ldots(ii)$$

where C_2 is another constant of integration. Thus two constants of integration (i.e., C_1 and C_2) are obtained from boundary conditions. The boundary conditions are :

(i) at $x = 0, y = 0$ and (ii) at $x = L, y = 0$

Substituting first boundary condition i.e., $x = 0, y = 0$ in equation (ii), we get

$$0 = 0 - 0 + 0 + C_2 \quad \text{or} \quad C_2 = 0$$

Substituting the second boundary condition i.e., at $x = L, y = 0$ in equation (ii), we get

$$0 = \frac{w \cdot L}{4} \cdot \frac{L^3}{3} - \frac{w}{6} \cdot \frac{L^4}{4} + C_1 \cdot L \qquad (C_2 \text{ is already zero})$$

$$= \frac{w \cdot L^4}{12} - \frac{w \cdot L^4}{24} + C_1 L$$

or

$$C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{24}$$

Substituting the value of C_1 in equations (i) and (ii), we get

$$EI \frac{dy}{dx} = \frac{w \cdot L}{4} \cdot x^2 - \frac{w}{6} x^3 - \frac{wL^3}{24} \qquad \ldots(iii)$$

and $$EI.y = \frac{w.L}{12}.x^3 - \frac{w}{24}.x^4 + \left(-\frac{wL^3}{24}\right)x + 0 \quad (\because C_2 = 0)$$

or $$EIy = \frac{w.L}{12}x^3 - \frac{w}{24}.x^4 - \frac{wL^3}{24}x \quad ...(iv)$$

Equation (*iii*) is known as *slope equation*. We can find the slope $\left(i.e., \text{ the value of } \frac{dy}{dx}\right)$ at any point on the beam by substituting the different values of x in this equation.

Equation (*iv*) is known as *deflection equation*. We can find the deflection (*i.e.*, the value of y) at any point on the beam by substituting the different values of x in this equation.

Slope at the Supports

Let θ_A = Slope at support A. This is equal to $\left(\frac{dy}{dx}\right)_{at\ A}$

and θ_B = Slop at support $B = \left(\frac{dy}{dx}\right)_{at\ B}$

At A, $x = 0$ and $\frac{dy}{dx} = \theta_A$

Substituting these values in equation (*iii*), we get

$$EI.\theta_A = \frac{wL}{4}\times 0 - \frac{w}{6}\times 0 - \frac{wL^3}{24}$$

$$= -\frac{wL^3}{24} = -\frac{WL^2}{24} \quad (\because w.L = W = \text{Total load})$$

$$\therefore \quad \theta_A = -\frac{WL^2}{24EI} \quad ...(12.12)$$

(Negative sign means that tangent at A makes an angle with AB in the anti-clockwise direction)

By symmetry, $$\theta_B = -\frac{WL^2}{24EI} \quad ...(12.13)$$

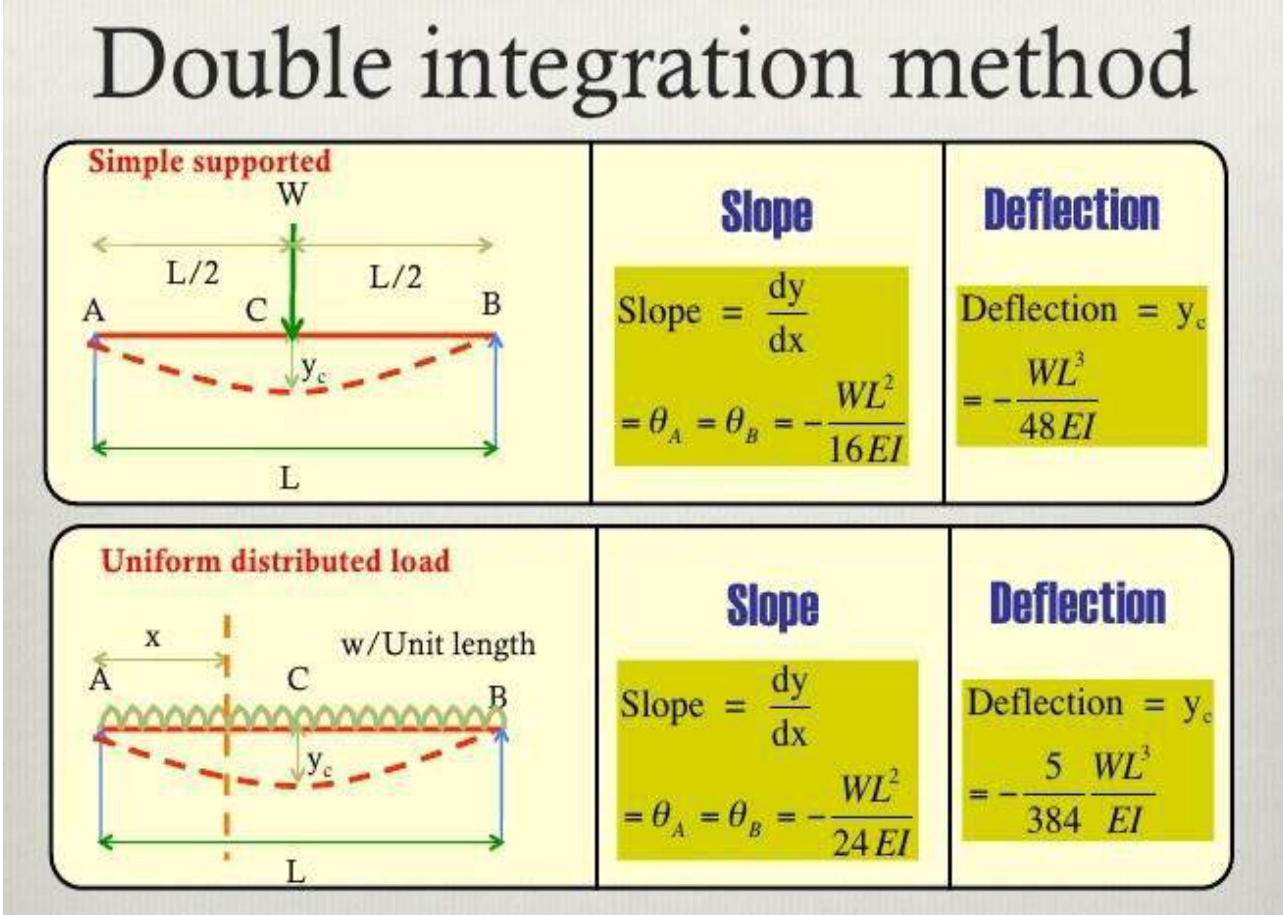
Maximum Deflection

The maximum deflection is at the centre of the beam *i.e.*, at point C, where $x = \frac{L}{2}$. Let y_C = deflection at C which is also maximum deflection. Substituting $y = y_C$ and $x = \frac{L}{2}$ in equation (*iv*), we get

$$EI.y_C = \frac{w.L}{12}\left(\frac{L}{2}\right)^3 - \frac{w}{24}\left(\frac{L}{2}\right)^4 - \frac{wL^3}{24}\left(\frac{L}{2}\right)$$

$$= \frac{w.L^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48} = -\frac{5w.L^4}{384}$$

$$\therefore \quad y_C = -\frac{5}{384}.\frac{wL^4}{EI} = -\frac{5}{384}.\frac{W.L^3}{EI} \quad (\because w.L = W = \text{Total load})$$



4. A beam of uniform rectangular section 200 mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 KN/m run over the entire span of 5 m. if the value of E for the beam material is 1 X 10^4 N/mm^2 , find the slope at the supports and maximum deflection.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm

w = 9 KN/m = 9000 N/m

E = 1 X 10^4 N/mm^2

b = 200 mm

d = 300 mm

SOLUTION:

1. SLOPE AT THE SUPPORTS,

$\theta_A = -WL^2 / 24EI$

$= \frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8}$

= **0.0104 radians.**

W = w.L = 9000 X 5 = 45000 N

I = $bd^3/12 = 200 \times 300^3 / 12$

$= 4.5 \times 10^8\ mm^4$

2. MAXIMUM DEFLECTION,

$$y = \frac{5\,W\,L^3}{384\,E\,I}$$

$$= \frac{5 \times 45000 \times 5000^3}{384 \times 1 \times 10^4 \times 4.5 \times 10^8}$$

= 16.27 mm.

5. A beam of length 5 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9 KN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm^2 and central deflection is not to exceed 1 cm.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm, w = 9 KN/m = 9000 N/m

∴ Total load, $W = w.L = 9 \times 5 = 45$ kN = 45000 N

Bending stress, $f = 7$ N/mm^2

Central deflection, $y_c = 1$ cm = 10 mm

Value of $E = 1 \times 10^4$ N/mm^2

Let b = Width of beam in mm

and d = Depth of beam in mm

∴ M.O.I., $I = \frac{bd^3}{12}$

Using equation (12.14), we get

$$y_C = \frac{5}{384} \cdot \frac{W \cdot L^3}{EI}$$

or
$$10 = \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times \left(\frac{bd^3}{12}\right)}$$

or
$$bd^3 = \frac{5}{384} \times \frac{45000 \times 5000^3 \times 12}{1 \times 10^4 \times 10}$$
$$= 878.906 \times 10^7 \text{ mm}^4 \quad \text{...(i)}$$

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by,

$$M = \frac{w \cdot L^2}{8} = \frac{W \cdot L}{8} \quad (\because W = w.L = \text{Total load})$$
$$= \frac{45000 \times 5}{8} \text{ Nm} = \frac{45000 \times 5}{8} \times 1000 \text{ Nmm}$$
$$= 28125000 \text{ Nmm}$$

Now using the bending equation as

$$\frac{M}{I} = \frac{f}{y}$$

or
$$\frac{28125000}{\left(\frac{bd^3}{12}\right)} = \frac{7}{\left(\frac{d}{2}\right)} \quad \left(\because \text{ Here } y = \frac{d}{2}\right)$$

or
$$\frac{28125000 \times 12}{bd^3} = \frac{14}{d}$$

or
$$bd^2 = \frac{28125000 \times 12}{14} = 24107142.85 \text{ mm}^2 \quad \text{...(ii)}$$

Dividing equation (i) by equation (ii), we get

$$d = \frac{878.906 \times 10^7}{24107142.85} = \mathbf{364.58 \text{ mm.}} \quad \textbf{Ans.}$$

Substituting this value of 'd' in equation (ii), we get

$$b \times (364.58)^2 = 24107142.85$$

∴
$$b = \frac{24107142.85}{364.58^2} = \mathbf{181.36 \text{ mm.}} \quad \textbf{Ans.}$$

Problem 12.7. *A beam of length 5 m and of uniform rectangular section is supported at its ends and carries uniformly distributed load over the entire length. Calculate the depth of the section if the maximum permissible bending stress is 8 N/mm² and central deflection is not to exceed 10 mm.*

Take the value of $E = 1.2 \times 10^4$ N/mm².

Sol. Given :

Length, $L = 5$ m $= 5000$ mm

Bending stress, $f = 8$ N/mm²

Central deflection, $y_c = 10$ mm

Value of $E = 1.2 \times 10^4$ N/mm²

Let W = Total load

and d = Depth of beam

The maximum bending moment for a simply supported beam carrying a uniformly distributed load is given by.

$$M = \frac{w.L^2}{8} = \frac{W.L}{8} \quad (\because W = w.L) \quad \text{...(i)}$$

Now using the bending equation,

$$\frac{M}{I} = \frac{f}{y}$$

or

$$M = \frac{f \times I}{y} = \frac{8 \times I}{(d/2)} \quad \left(\because y = \frac{d}{2}\right)$$

$\therefore$

$$M = \frac{16I}{d} \quad \text{...(ii)}$$

Equating the two values of B.M., we get

$$\frac{W.L}{8} = \frac{16I}{d}$$

or

$$W = \frac{16 \times 8I}{L \times d} = \frac{128I}{L \times d} \quad \text{...(iii)}$$

Now using equation (12.14), we get

$$y_C = \frac{5}{384} \times \frac{WL^3}{EI}$$

or

$$10 = \frac{5}{384} \times \frac{128I}{L \times d} \times \frac{L^3}{EI} \quad \left(\because y_C = 10 \text{ mm and } W = \frac{128I}{L \times d}\right)$$

$$= \frac{5}{384} \times \frac{128 \times L^2}{d \times E}$$

or

$$d = \frac{5}{384} \times \frac{128 \times L^2}{10 \times E} = \frac{5}{384} \times \frac{128 \times 5000^2}{10 \times 1.2 \times 10^4}$$

$= 347.2$ mm $=$ **34.72 cm. Ans.**

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH AN ECCENTRIC POINT LOAD

- SLOPE AT THE LEFT SUPPORT,

$$\theta_A = \frac{-W.a.b}{6EI.L}(a + 2b)$$

- MAXIMUM DEFLECTION,

$$y_{max} = \frac{W.b}{9\sqrt{3}EI.L}(a^2 + 2ab)^{3/2}$$

- DEFLECTION UNDER THE POINT LOAD,

$$y_C = \frac{Wa^2b^2}{3EIL}$$

6. Determine slope at the left support, deflection under the load and maximum deflection of a simply supported beam of length 5 m, which is carrying a point load of 5 KN at a distance of 3 m from the left end. Take E = 2 X 10^5 N/mm^2 and I = 1 X 10^8 mm^4.

GIVEN DATA:

L = 5 m = 5 X 10^3 mm

W = 5 KN = 5 X 10^3 N

I = 1 X 10^8 mm^4.

E = 2 X 10^5 N/mm^2

a = 3 m

b = L – a = 5 – 3 = 2 m = 2 X 10^3 mm

SOLUTION:

1. SLOPE AT THE LEFT SUPPORT,

$$\theta_A = \frac{-W.a.b}{6EI.L}(a+2b)$$

= 0.00035 radians.

2. DEFLECTION UNDER THE POINT LOAD,

$$y_C = \frac{Wa^2b^2}{3EIL}$$

= 0.6 mm.

3. MAXIMUM DEFLECTION,

$$y_{max} = \frac{W.b}{9\sqrt{3}EI.L}(a^2 + 2ab)^{3/2}$$

= 0.6173 mm.

tion is expressed and in the manner in which the integrations are carried out.

12.7.1. Deflection of a Simply Supported Beam with an Eccentric Point Load. A simply supported beam *AB* of length *L* and carrying a point load *W* at a distance '*a*' from left support and at a distance '*b*' from right support is shown in Fig. 12.7. The reactions at *A* and *B* are given by,

$$R_A = \frac{W.b}{L} \quad \text{and} \quad R_B = \frac{W.a}{L}$$

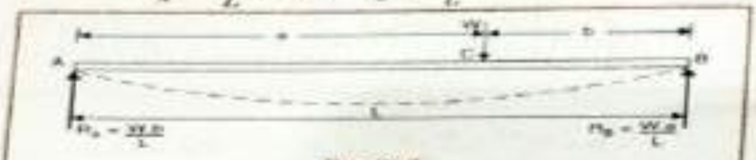


Fig. 12.7

The bending moment at any section between *A* and *C* at a distance *x* from *A* is given by,

$$M_x = R_A \times x = \frac{W.b}{L} \times x$$

The above equation of B.M. holds good for the values of *x* between 0 and '*a*'. The B.M. at any section between *C* and *B* at a distance *x* from *A* is given by,

$$M_x = R_A . x - W \times (x - a)$$

$$= \frac{W.b}{L} . x - W(x - a)$$

The above equation of B.M. holds good for all values of *x* between *x* = *a* and *x* = *b*.

The B.M. for all sections of the beam can be expressed in a single equation written as

$$M_x = \frac{W.b}{L} x \quad \vdots \quad - W(x - a) \qquad ...(i)$$

Stop at the dotted line for any point in section *AC*. But for any point in section *CB*, add the expression beyond the dotted line also.

The B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2} \qquad ...(ii)$$

Hence equating (*i*) and (*ii*), we get

$$EI \frac{d^2y}{dx^2} = \frac{W.b}{L} . x \quad \vdots \quad - W(x - a) \qquad ...(iii)$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{W.b}{L} \frac{x^2}{2} + C_1 \quad \vdots \quad - \frac{W(x - a)^2}{2} \qquad ...(iv)$$

where C_1 is a constant of integration. This constant of integration should be written after the first term. Also *the brackets are to be integrated as a whole*. Hence the integration of $(x - a)$ will be $\frac{(x-a)^2}{2}$ and not $\frac{x^2}{2} - ax$.

Integrating equation (iv) once again, we get

$$EIy = \frac{W.b}{2L}.\frac{x^3}{3} + C_1x + C_2 \quad \vdots \quad - \frac{W}{2}\frac{(x-a)^3}{3} \qquad ...(v)$$

where C_2 is another constant of integration. This constant is written after C_1x. The integration of $(x-a)^2$ will be $\left(\frac{x-a}{3}\right)^3$. This type of integration is justified as the constant of integrations C_1 and C_2 are valid for all values of x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i) At $x = 0$, $y = 0$ and (ii) At $x = L$, $y = 0$

(i) At A, $x = 0$ and $y = 0$. Substituting these values in equation (v) upto dotted line only, we get

$$0 = 0 + 0 + C_2$$

$$\therefore \quad C_2 = 0$$

(ii) At B, $x = L$ and $y = 0$. Substituting these values in equation (v), we get

$$0 = \frac{W.b}{2L}.\frac{L^3}{3} + C_1 \times L + 0 - \frac{W}{2}\frac{(L-a)^3}{3}$$

($\because$ $C_2 = 0$. Here complete Eq. (v) is to be taken)

$$= \frac{W.b.L^2}{6} + C_1 \times L - \frac{W}{2}\frac{b^3}{3} \qquad (\because L - a = b)$$

$$\therefore \quad C_1 \times L = \frac{W}{6}.b^3 - \frac{W.b.L^2}{6} = -\frac{W.b}{6}(L^2 - b^2)$$

$$\therefore \quad C_1 = -\frac{W.b}{6L}(L^2 - b^2) \qquad ...(vi)$$

Substituting the value of C_1 in equation (iv), we get

$$EI\frac{dy}{dx} = \frac{W.b}{L}\frac{x^2}{2} + \left[-\frac{W.b}{6L}(L^2 - b^2)\right] \quad \vdots \quad - \frac{W(x-a)^2}{2}$$

$$= \frac{W.b.x^2}{2L} - \frac{W.b}{6L}(L^2 - b^2) \quad \vdots \quad - \frac{W(x-a)^2}{2} \qquad ...(vii)$$

Equation (vii) gives the slope at any point in the beam. Slope is maximum at A or B. To find the slope at A, substitute $x = 0$ in the above equation upto dotted line as point A lies in AC.

$$\therefore \quad EI.\theta_A = \frac{W.b}{2L} \times 0 - \frac{Wb}{6L}(L^2 - b^2) \qquad \left(\because \frac{dy}{dx} \text{ at } A = \theta_A\right)$$

$$= -\frac{Wb}{6L}(L^2 - b^2)$$

$$\therefore \quad \theta_A = -\frac{Wb}{6EIL}(L^2 - b^2) \qquad \text{(as given before)}$$

Substituting the values of C_1 and C_2 in equation (v), we get

$$EIy = \frac{W.b}{6L}x^3 + \left[-\frac{Wb}{6L}(L^2 - b^2)\right]x + 0 \quad \vdots \quad - \frac{W}{6}(x-a)^3 \qquad ...(viii)$$

Equation (viii) gives the deflection at any point in the beam. To find the deflection y_c under the load, substitute $x = a$ in equation (viii) and consider the equation upto dotted line (as point C lies in AC). Hence, we get

$$EIy_c = \frac{W.b}{6L}.a^3 - \frac{W.b}{6L}(L^2 - b^2)a = \frac{W.b}{6L}.a(a^2 - L^2 + b^2)$$

$$= -\frac{W.a.b}{6L}(L^2 - a^2 - b^2)$$

$$= -\frac{W.a.b}{6L}[(a+b)^2 - a^2 - b^2] \qquad (\because L = a + b)$$

$$= -\frac{W.a.b}{6L}[a^2 + b^2 + 2ab - a^2 - b^2]$$

$$= -\frac{W.a.b}{6L}[2ab] = -\frac{Wa^2.b^2}{3L}$$

$$\therefore \quad y_c = -\frac{Wa^2.b^2}{3EIL} \qquad \text{...(same as before)}$$

Note. While using Macaulay's Method, the section x is to be taken in the last portion of the beam.

Problem 12.8. *A beam of length 6 m is simply supported at its ends and carries a point load of 40 kN at a distance of 4 m from the left support. Find the deflection under the load and maximum deflection. Also calculate the point at which maximum deflection takes place. Given M.O.I. of beam = 7.33×10^7 mm^4 and $E = 2 \times 10^5$ N/mm^2.*

Sol. Given :

Length, $L = 6$ m $= 6000$ mm

Point load, $W = 40$ kN $= 40,000$ N

Distance of point load from left support, $a = 4$ m $= 4000$ mm

$\therefore$ $b = L - a = 6 - 4 = 2$ m $= 2000$ mm

Let y_c = Deflection under the load

y_{max} = Maximum deflection

Using equation $y_c = -\frac{W.a^2.b^2}{3EIL}$

$$\therefore \quad y_c = -\frac{40000 \times 4000^2 \times 2000^2}{3 \times 2 \times 10^5 \times 7.33 \times 10^7 \times 6000}$$

$$= -9.7 \text{ mm. } \textbf{Ans.}$$

Problem 12.9. *A beam of length 6 m is simply supported at its ends and carries two point loads of 48 kN and 40 kN at a distance of 1 m and 3 m respectively from the left support. Find :*

(i) deflection under each load,

(ii) maximum deflection, and

(iii) the point at which maximum deflection occurs.

Given $E = 2 \times 10^5$ N/mm^2 and $I = 85 \times 10^6$ mm^4.

Sol. Given :

$I = 85 \times 10^6$ mm^4 ; $E = 2 \times 10^5$ N/mm^2

First calculate the reactions R_A and R_B.

Taking moments about A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3 = 168$$

$$\therefore \quad R_B = \frac{168}{6} = 28 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = (48 + 40) - 28 = 60 \text{ kN}$$

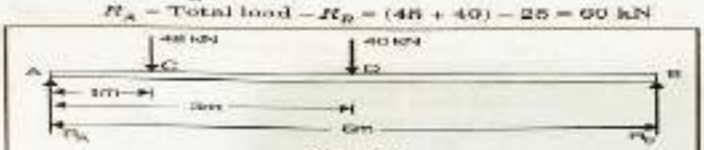


Fig. 12.9

Consider the section X in the last part of the beam (*i.e.*, in length DB) at a distance x from the left support A. The B.M. at this section is given by,

$$EI \frac{d^2y}{dx^2} = R_A.x \;\vdots\; - 48(x - 1) \;\vdots\; - 40(x - 3)$$

$$= 60x \;\vdots\; - 48(x - 1) \;\vdots\; - 40(x - 3)$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; - 48 \frac{(x-1)^2}{2} \;\vdots\; - 40 \frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; - 24(x - 1)^2 \;\vdots\; - 20(x - 3)^2 \quad ...(i)$$

Integrating the above equation again, we get

$$EIy = \frac{30x^3}{3} + C_1x + C_2 \;\vdots\; \frac{-24(x-1)^3}{3} \;\vdots\; \frac{-20(x-3)^3}{3}$$

$$= 10x^3 + C_1x + C_2 \;\vdots\; - 8(x - 1)^3 \;\vdots\; - \frac{20}{3}(x - 3)^3 \quad ...(ii)$$

To find the values of C_1 and C_2, use two boundary conditions. The boundary conditions are:

(i) at $x = 0$, $y = 0$, and (ii) at $x = 6$ m, $y = 0$.

(i) Substituting the first boundary condition *i.e.*, at $x = 0$, $y = 0$ in equation (ii) and considering the equation upto first dotted line (as $x = 0$ lies in the first part of the beam), we get

$$0 = 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(ii) Substituting the second boundary condition *i.e.*, at $x = 6$ m, $y = 0$ in equation (ii) and considering the complete equation (as $x = 6$ lies in the last part of the beam), we get

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3 \quad (\because C_2 = 0)$$

or

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore \quad C_1 = \frac{-980}{6} = -163.33$$

Now substituting the values of C_1 and C_2 in equation (ii), we get

$$EIy = 10x^3 - 163.33x \;\vdots\; - 8(x - 1)^3 \;\vdots\; - \frac{20}{3}(x - 3)^3 \quad ...(iii)$$

(i) (a) *Deflection under first load i.e., at point C.* This is obtained by substituting $x = 1$ in equation (iii) upto the first dotted line (as the point C lies in the first part of the beam). Hence, we get

$$EI.y_c = 10 \times 1^3 - 163.33 \times 1$$

$$= 10 - 163.33 = -153.33 \text{ kNm}^3$$

$$= -153.33 \times 10^3 \text{ Nm}^3$$

$$= -153.33 \times 10^3 \times 10^9 \text{ Nmm}^3$$

$$= -153.33 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_c = \frac{-153.33 \times 10^{12}}{EI} = \frac{-153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$= -9.019 \text{ mm.} \quad \textbf{Ans.}$$

(Negative sign shows that deflection is downwards).

(b) *Deflection under second load i.e. at point D.* This is obtained by substituting $x = 3$ m in equation (iii) upto the second dotted line (as the point D lies in the second part of the beam). Hence, we get

$$EI.y_D = 10 \times 3^3 - 163.33 \times 3 - 8(3 - 1)^3$$

$$= 270 - 489.99 - 64 = -283.99 \text{ kNm}^3$$

$$= -283.99 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_D = \frac{-283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -16.7 \text{ mm.} \quad \textbf{Ans.}$$

(ii) *Maximum Deflection.* The deflection is likely to be maximum at a section between C and D. For maximum deflection, $\frac{dy}{dx}$ should be zero. Hence equate the equation (i) equal to zero upto the second dotted line.

$$\therefore \quad 30x^2 + C_1 - 24(x - 1)^2 = 0$$

or $$30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0 \quad (\because C_1 = -163.33)$$

or $$6x^2 + 48x - 187.33 = 0$$

The above equation is a quadratic equation. Hence its solution is

$$x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} = 2.87 \text{ m.}$$

(Neglecting – ve root)

Now substituting $x = 2.87$ m in equation (iii) upto the second dotted line, we get maximum deflection as

$$EIy_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87 - 1)^3$$

$$= 236.39 - 468.75 - 52.31$$

$$= -284.67 \text{ kNm}^3 = -284.67 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -16.745 \text{ mm.} \quad \textbf{Ans.}$$

MOMENT AREA METHOD:

✓ **MOHR'S THEOREM – I:**

The change of slope between any two points is equal to the net area of the B.M. diagram between these points divided by EI.

✓ **MOHR'S THEOREM – II:**

The total deflection between any two points is equal to the moment of the area of B.M. diagram between the two points about the last point divided by EI.

MOHR'S THEOREMS IS USED FOR FOLLOWING CASES:

- ✓ Problems on Cantilevers
- ✓ Simply supported beams carrying symmetrical loading
- ✓ Fixed beams

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

Fig. 12.19 (a) shows a simply supported AB of length L and carrying a point load W at the centre of the beam *i.e.*, at point C. The B.M. diagram is shown in Fig. 12.19 (b). This is a case of symmetrical loading, hence slope is zero at the centre *i.e.*, at point C.

But the deflection is maximum at the centre.

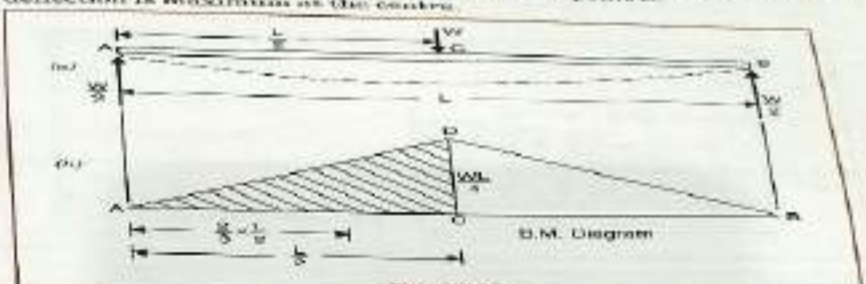


Fig. 12.19

Now using Mohr's theorem for slope, we get

Slope at $A = \frac{\text{Area of B.M. diagram between A and C}}{EI}$

But area of B.M. diagram between A and C

$= \text{Area of triangle } ACD$

$= \frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4} = \frac{WL^2}{16}$

$\therefore$ Slope at A or $\theta_A = \frac{WL^2}{EI}$

Now using Mohr's theorem for deflection, we get from equation (12.17) as

$$y = \frac{A\bar{x}}{EI}$$

where A = Area of B.M. Diagram between A and C

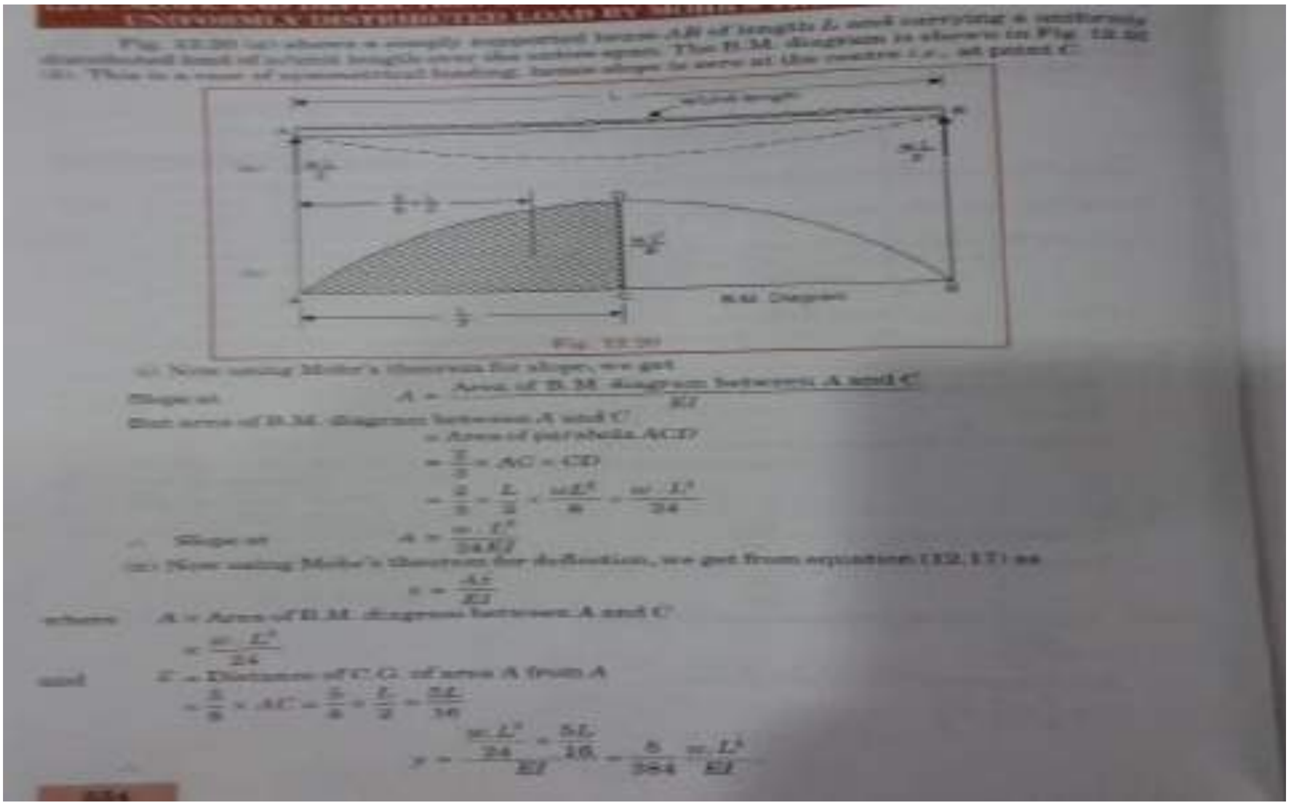
$= \frac{WL^2}{16}$

$\bar{x}$ = Distance of C.G. of area A from A

$= \frac{2}{3} \times \frac{L}{2} = \frac{L}{3}$

$$\therefore \quad y = \frac{\frac{WL^2}{16} \times \frac{L}{3}}{EI} = \frac{WL^3}{48EI}.$$

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED WITH A UNIFORMLY DISTRIBUTED LOAD:



(i) Now using Mohr's theorem for slope, we get

Slope at $A = \frac{\text{Area of B.M. diagram between A and C}}{EI}$

But area of B.M. diagram between A and C

$= \text{Area of parabola } ACD$

$= \frac{2}{3} \times AC \times CD$

$= \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{w \cdot L^3}{24}$

$\therefore$ Slope at $A = \frac{w \cdot L^3}{24EI}$

(ii) Now using Mohr's theorem for deflection, we get from equation (12.17) as

$y = \frac{A\bar{x}}{EI}$

where A = Area of B.M. diagram between A and C

$= \frac{w \cdot L^3}{24}$

and $\bar{x}$ = Distance of C.G. of area A from A

$= \frac{5}{8} \times AC = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}$

$y = \frac{\frac{w \cdot L^3}{24} \times \frac{5L}{16}}{EI} = \frac{5}{384} \frac{w \cdot L^4}{EI}$

CONJUGATE BEAM METHOD:

- **CONJUGATE BEAM:**
  - ✓ Conjugate beam is an imaginary beam of length equal to that of the original beam but for which the load diagram is the M/EI diagram.
  - **NOTE 1 :**
  - ✓ The slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.
  - **NOTE 2 :**
  - ✓ The deflection at any section for the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

SLOPE AND DEFLECTION FOR A SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD:

- ✓ A simply supported beam AB of length L carrying a point load W at the centre C.
- ✓ The B.M at A and B is zero and at the centre B.M will be WL/4.
- ✓ Now the conjugate beam AB can be constructed.

- ✓ The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.
- ✓ The shape of the loading on the conjugate beam will be same as of B.M diagram.
- ✓ The ordinate of loading on conjugate beam will be equal to M/EI = WL/4EI.

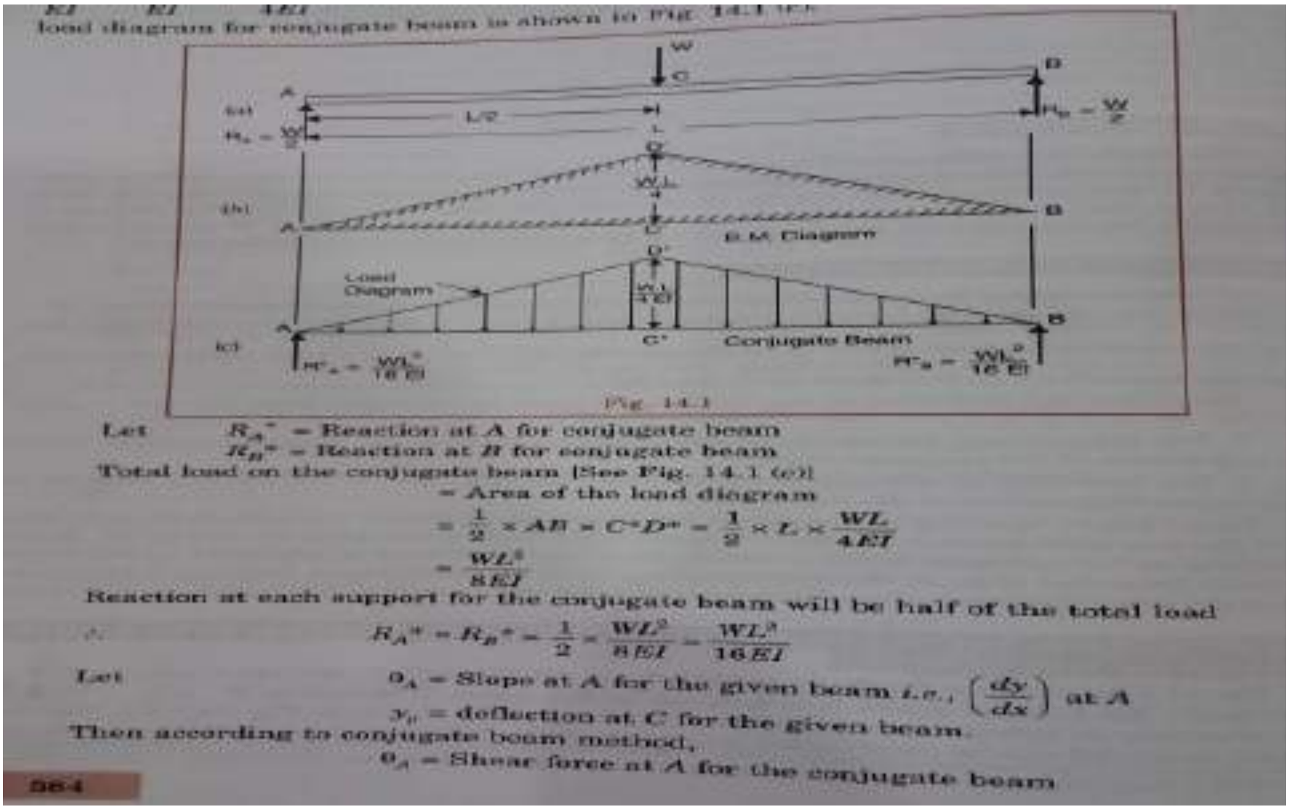

load diagram for conjugate beam is shown in Fig. 14.1 (c).

Fig. 14.1

Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam [See Fig. 14.1 (c)]

$$= \text{Area of the load diagram}$$

$$= \frac{1}{2} \times AB \times C^*D^* = \frac{1}{2} \times L \times \frac{WL}{4EI}$$

$$= \frac{WL^2}{8EI}$$

Reaction at each support for the conjugate beam will be half of the total load

$$\therefore \quad R_A^* = R_B^* = \frac{1}{2} \cdot \frac{WL^2}{8EI} = \frac{WL^2}{16EI}$$

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at A

y_c = deflection at C for the given beam.

Then according to conjugate beam method,

θ_A = Shear force at A for the conjugate beam

564

$$= R_A^* \qquad (\because \text{ S.F. at } A \text{ for conjugate beam} = R_A^*)$$

$$= \frac{WL^2}{16EI}$$

$$y_C = \text{B.M. at } C \text{ for the conjugate beam} \qquad \text{[See Fig. 14.1 (c)]}$$

$$= R_A^* \times \frac{L}{2} - \text{Load corresponding to } AC^*D^* \times \text{Distance of C.G. of } AC^*D^* \text{ from } C$$

$$= \frac{WL^2}{16EI} \cdot \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) \times \left(\frac{1}{3} \times \frac{L}{2}\right)$$

$$= \frac{WL^3}{32EI} - \frac{WL^3}{96EI} = \frac{3WL^3 - WL^3}{96EI}$$

$$= \frac{WL^3}{48EI}.$$

$6EIL$

Problem 14.1. *A simply supported beam of length 5 m carries a point load of 5 kN at a distance of 3 m from the left end. If E = 2 × 10⁵ N/mm² and I = 10⁸ mm⁴, determine the slope at the left support and deflection under the point load using conjugate beam method.*

Sol. Given :

Length, $L = 5$ m

Point load, $W = 5$ kN

Distance AC, $a = 3$ m

Distance BC, $b = 5 - 3 = 2$ m

Value of $E = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^5 \times 10^6 \text{ N/m}^2$
$= 2 \times 10^5 \times 10^3 \text{ kN/m}^2 = 2 \times 10^8 \text{ kN/m}^2$

Value of $I = 1 \times 10^8 \text{ mm}^4 = 10^{-4} \text{ m}^4$

Let R_A = Reaction at A

and R_B = Reaction at B.

Taking moments about A, we get

$$R_B \times 5 = 5 \times 3$$

$$\therefore \quad R_B = \frac{5 \times 3}{5} = 3 \text{ kN}$$

and $R_A = \text{Total load} - R_B = 5 - 3 = 2$ kN

The B.M. at $A = 0$

B.M. at $B = 0$

B.M. at $C = R_A \times 3 = 2 \times 3 = 6$ kNm.

Now B.M. diagram is drawn as shown in Fig. 14.3 (b).

Now construct the conjugate beam as shown in Fig. 14.3 (c). The vertical load at C^* on conjugate beam

$$= \frac{\text{B.M. at } C}{EI} = \frac{6 \text{ kNm}}{EI}$$

Now calculate the reaction at A^* and B^* for conjugate beam

Let R_A^* = Reaction at A^* for conjugate beam

R_B^* = Reaction at B^* for conjugate beam.

Taking moments about A^*, we get

$$R_B^* \times 5 = \text{Load on } A^*C^*D^* \times \text{distance of C.G. of } A^*C^*D^* \text{ from } A^* + \text{Load on } B^*C^*D^* \times \text{Distance of C.G. of } B^*C^*D^* \text{ from } A^*$$

$$= \left(\frac{1}{2} \times 3 \times \frac{6}{EI}\right) \times \left(\frac{2}{3} \times 3\right) + \left(\frac{1}{2} \times 2 \times \frac{6}{EI}\right) \times \left(3 + \frac{1}{3} \times 2\right)$$

$$= \frac{18}{EI} + \frac{6}{EI} \times \frac{11}{3} = \frac{8}{EI} + \frac{22}{EI} = \frac{40}{EI}$$

$$\therefore \quad R_B^* = \frac{40}{EI} \times \frac{1}{5} = \frac{8}{EI}$$

587

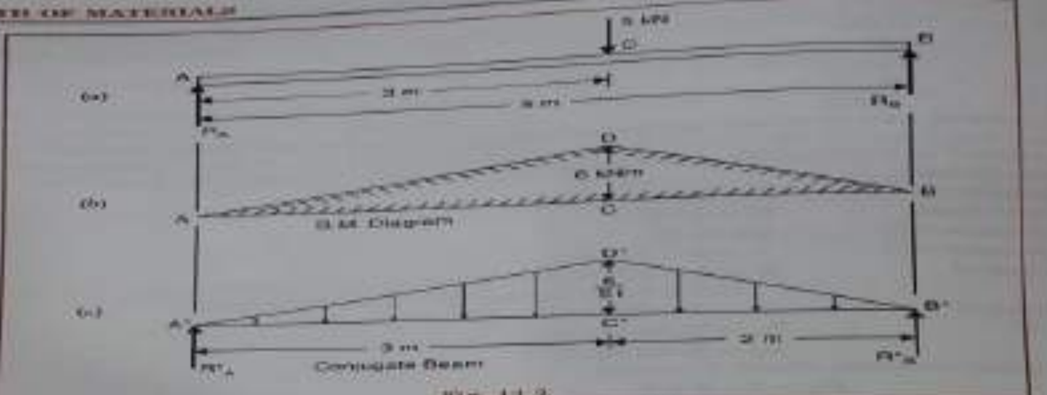


Fig. 14.3

$$R_A^* = \text{Total load } (i.e., \text{ load } A^*B^*D^*) - R_B^*$$
$$= \left(\frac{1}{2} \times 5 \times \frac{6}{EI}\right) - \frac{8}{EI}$$
$$= \frac{15}{EI} - \frac{8}{EI} = \frac{7}{EI}$$

Let θ_A = Slope at A for the given beam i.e., $\left(\frac{dy}{dx}\right)$ at A

y_C = Deflection at C for the given beam

Then according to conjugate beam method,

θ_A = Shear force at A^* for conjugate beam $= R_A^*$

$$= \frac{7}{EI} = \frac{7}{2 \times 10^8 \times 10^{-4}} \quad (\because E = 2 \times 10^8 \text{ kN/m}^2 \text{ and } I = 10^{-4} \text{ m}^4)$$

= **0.00035 radians. Ans.**

y_C = B.M. at C^* for conjugate beam

$= R_A^* \times 3$ − Load $A^*C^*D^*$ × Distance of C.G. of $A^*C^*D^*$ from C^*

$$= \frac{7}{EI} \times 3 - \left(\frac{1}{2} \times 3 \times \frac{6}{EI}\right) \times \left(\frac{1}{3} \times 3\right)$$
$$= \frac{21}{EI} - \frac{9}{EI} = \frac{12}{EI}$$
$$= \frac{12}{2 \times 10^8 \times 10^{-4}} = \frac{6}{10^4} \text{ m} = \frac{6 \times 1000}{10000} \text{ mm} = \mathbf{0.6 \text{ mm.}} \text{ Ans.}$$

Problem 14.2. *A simply supported beam of length 4 m carries a point load of 3 kN at a distance of 1 m from each end. If E = 2 × 10⁵ N/mm² and I = 10⁸ mm⁴ for the beam, then using conjugate beam method determine : (i) slope at each end and under each load (ii) deflection under each load and at the centre.*

Sol. Given :

Length, $L = 4$ m

Value of $E = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^5 \times 10^6 \text{ N/m}^2$
$= 2 \times 10^5 \times 10^3 \text{ kN/m}^2 = 2 \times 10^8 \text{ kN/m}^2$

Value of $I = 10^8 \text{ mm}^4 = \frac{10^8}{10^{12}} \text{ m}^4 = 10^{-4} \text{ m}^4$

As the load on the beam is symmetrical as shown in Fig. 14.4 (a), the reactions R_A and R_B will be equal to 3 kN.

Now B.M. at A and B are zero.

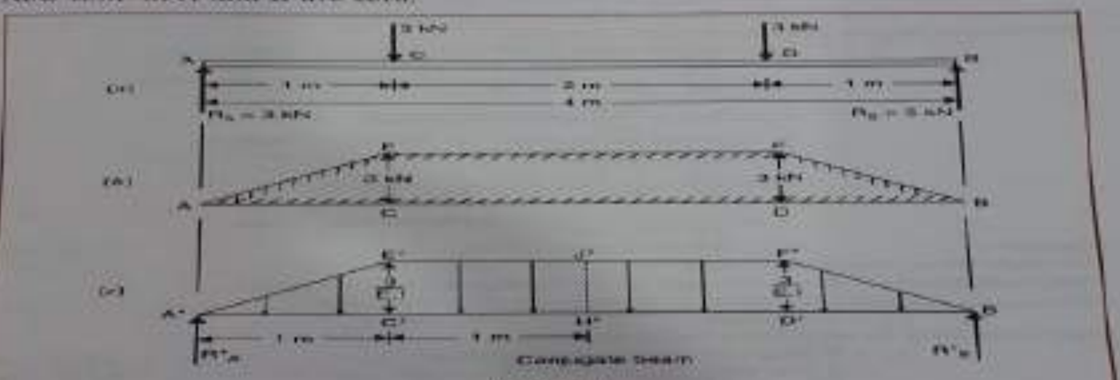


Fig. 14.4

B.M. at $C = R_A \times 1 = 3 \times 1 = 3$ kNm

B.M. at $D = R_B \times 1 = 3 \times 1 = 3$ kNm

Now B.M. diagram can be drawn as shown in Fig. 14.4 (b).

Now by dividing the B.M. at any section by EI, we can construct the conjugate beam as shown in Fig. 14.4 (c). The loading are shown on the conjugate beam.

Let R_A^* = Reaction at A^* for the conjugate beam and

R_B^* = Reaction at B^* for conjugate beam

The loading on the conjugate beam is symmetrical

$\therefore \quad R_A{}^* = R_B{}^* =$ Half of total load on conjugate beam

$= \frac{1}{2}$ [Area of trapezoidal $A^*B^*F^*E^*$]

$$= \frac{1}{2}\left[\frac{(E^*F^* + A^*B^*)}{2} \times E^*C^*\right]$$

$$= \frac{1}{2}\left[\frac{(2+4)}{2} \times \frac{3}{EI}\right] = \frac{4.5}{EI}$$

(i) Slope at each end and under each load

Let θ_A = Slope at A for the given beam *i.e.*, $\left(\frac{dy}{dx}\right)$ at **A**

θ_B = Slope at B for the given beam

θ_C = Slope at C for the given beam and

θ_D = Slope at D for the given beam

Then according to conjugate beam method,

θ_A = Shear force at A^* for conjugate beam $= R_A{}^*$

$$= \frac{4.5}{EI} = \frac{4.5}{2 \times 10^8 \times 10^{-4}} = \mathbf{0.000225\ rad.} \quad \textbf{Ans.}$$

$$\theta_B = R_B{}^* = \frac{4.5}{EI} = \mathbf{0.000225\ rad.} \quad \textbf{Ans.}$$

θ_C = Shear force at C^* for conjugate beam

$= R_A{}^*$ − Total load $A^*C^*D^*$

$$= \frac{4.5}{EI} - \frac{1}{2} \times 1 \times \frac{3}{EI} = \frac{3}{EI}$$

$$= \frac{3}{2 \times 10^8 \times 10^{-4}} = \mathbf{0.00015\ rad.} \quad \textbf{Ans.}$$

Similarly, θ_D = **0.00015 rad. Ans.** (By symmetr

13.1. INTRODUCTION

Cantilever is a beam whose one end is fixed and other end is free. In this chapter we shall discuss the methods of finding slope and deflection for the cantilevers when they are subjected to various types of loading. The important methods are (i) Double integration method (ii) Macaulay's method and (iii) Moment-area-method. These methods have also been used for finding deflections and slope of the simply supported beams.

13.2. DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a point load at the free end B is shown in Fig. 13.1. AB shows the position of cantilever before any load is applied whereas AB' shows the position of the cantilever after loading.

Fig. 13.1

Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -W(L-x) \qquad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation (12.3) as

$$M = EI\frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = -W(L-x) = -WL + W.x$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C_1 \qquad ...(i)$$

Integrating again, we get

$$EIy = -WL\frac{x^2}{2} + \frac{W}{2}\frac{x^3}{3} + C_1x + C_2 \qquad ...(ii)$$

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (i) at $x = 0, y = 0$ (ii) $x = 0, \frac{dy}{dx} = 0$

[At the fixed end, deflection and slopes are zero]

(i) By substituting $x = 0, y = 0$ in equation (ii), we get

$$0 = 0 + 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(ii) By substituting $x = 0, \frac{dy}{dx} = 0$ in equation (i), we get

$$0 = 0 + 0 + C_1 \quad \therefore \quad C_1 = 0$$

Substituting the value of C_1 in equation (i), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left(Lx - \frac{x^2}{2}\right) \qquad ...(iii)$$

Equation (iii) is known as *slope equation*. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting $x = L$ in these equations.

Substituting the values of C_1 and C_2 in equation (ii), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6} \qquad (\because C_1 = 0, C_2 = 0)$$

$$= -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \qquad ...(iv)$$

Equation (iv) is known as *deflection equation*.

Let θ_B = slope at the free end B i.e., $\left(\frac{dy}{dx}\right)$ at $B = \theta_B$ and

y_B = Deflection at the free end B

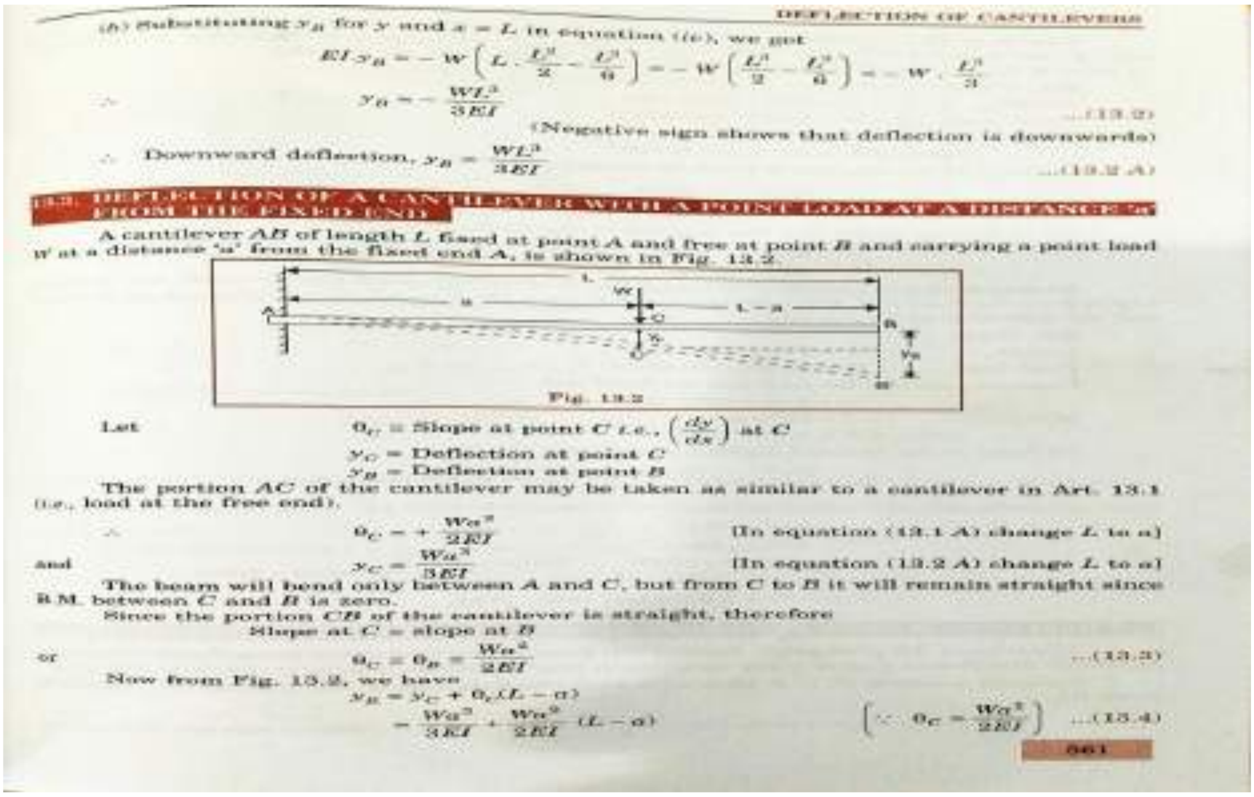
(a) Substituting θ_B for $\frac{dy}{dx}$ and $x = L$ in equation (iii), we get

$$EI.\theta_B = -W\left(L.L - \frac{L^2}{2}\right) = -W.\frac{L^2}{2}$$

$$\therefore \quad \theta_B = -\frac{WL^2}{2EI} \qquad ...(13.1)$$

Negative sign shows that tangent at B makes an angle in the anti-clockwise direction with AB

$$\therefore \quad \theta_B = \frac{WL^2}{2EI} \qquad ...(13.1A)$$



DEFLECTION OF CANTILEVERS

(b) Substituting y_B for y and $x = L$ in equation (iv), we get

$$EI.y_B = -W\left(L.\frac{L^2}{2}-\frac{L^3}{6}\right) = -W\left(\frac{L^3}{2}-\frac{L^3}{6}\right) = -W.\frac{L^3}{3}$$

$\therefore \quad y_B = -\frac{WL^3}{3EI}$ ...(13.2)

(Negative sign shows that deflection is downwards)

$\therefore$ Downward deflection, $y_B = \frac{WL^3}{3EI}$ ...(13.2 A)

13.3 DEFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig. 13.2.

Fig. 13.2

Let θ_C = Slope at point C i.e., $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. 13.1 (i.e., load at the free end).

$\therefore \quad \theta_C = +\frac{Wa^2}{2EI}$ [In equation (13.1 A) change L to a]

and $\quad y_C = \frac{Wa^3}{3EI}$ [In equation (13.2 A) change L to a]

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = slope at B

or $\quad \theta_C = \theta_B = \frac{Wa^2}{2EI}$ ...(13.3)

Now from Fig. 13.2, we have

$y_B = y_C + \theta_C(L-a)$

$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a)$ $\left[\because \theta_C = \frac{Wa^2}{2EI}\right]$ ...(13.4)

561

PROBLEMS:

1.A cantilever of length 3 m is carrying a point load of 25 KN at the free end. If I = 10^8 mm^4 and E = 2.1 X 10^5 N/mm^2, find the slope and deflection at the free end.

GIVEN DATA:

L = 3 m = 3000 mm

W = 25 KN = 25000 N

I = 10^8 mm^4

E = 2.1 X 10^5 N/mm^2

SOLUTION:

1. SLOPE AT THE FREE END,

$$\theta_B = WL^2 / 2EI = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8}$$

= 0.005357 radians.

2. DEFLECTION AT THE FREE END,

$$y_B = WL^3 / 3EI = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8}$$

= 10.71 mm

2. A cantilever of length 3 m is carrying a point load of 50 KN at a distance of 2 m from the fixed end. If I = 10^8 mm^4 and E = 2 X 10^5 N/mm^2, find the slope and deflection at the free end.

GIVEN DATA:

L = 3 m = 3000 mm

W = 50 KN = 50000 N

I = 10^8 mm^4

E = 2 X 10^5 N/mm^2

SOLUTION:

1. SLOPE AT THE FREE END,

θ_B = $Wa^2 / 2\,EI$

$$= \frac{50000 \times 2000}{2 \times 2 \times 10^5 \times 10^8}$$

= 0.005 radians

2. DEFLECTION AT THE FREE END,

y_B = $W a^3 / 3\,EI + W a^2 / 2\,EI\,(L - a)$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8}\,(3000 - 2000)$$

= 6.67 + 5

= 11.67 mm.

CANTILEVER BEAM WITH A UDL:

- A cantilever beam AB of length L fixed at the point A and free at the point B and carrying a UDL of w per unit length over the whole length.
- Consider a section X, at a distance x from the fixed end A.
- The bending moment at this section is given by,

$$M_x = \frac{-\,w\,(L - x)\,(L - x)}{2}$$

Fig. 13.3

But B.M. at any section is also given by equation (12.3) as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = -\frac{w}{2}(L-x)^2$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = -\frac{w}{2} \frac{(L-x)^3}{3}(-1) + C_1$$

$$= \frac{w}{6}(L-x)^3 + C_1 \quad \text{...(i)}$$

Integrating again, we get

$$EIy = \frac{w}{6} \cdot \frac{(L-x)^4}{4}(-1) + C_1x + C_2$$

$$= -\frac{w}{24}(L-x)^4 + C_1x + C_2 \quad \text{...(ii)}$$

Their values are obtained from boundary condi-

...(ii)

where C_1 and C_2 are constant of integrations. Their values are obtained from boundary conditions, which are : (*i*) at $x = 0$, $y = 0$ and (*ii*) at $x = 0$, $\frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting $x = 0$, $y = 0$ in equation (*ii*), we get

$$0 = -\frac{w}{24}(L-0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \quad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting $x = 0$ and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$0 = \frac{w}{6}(L-0)^3 + C_1 = \frac{wL^3}{6} + C_1$$

$$\therefore \quad C_1 = -\frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equations (i) and (ii), we get

$$EI\frac{dy}{dx} = \frac{w}{6}(L-x)^3 - \frac{wL^3}{6} \quad ...(iii)$$

and

$$EIy = -\frac{w}{24}(L-x)^4 - \frac{wL^3}{6}x + \frac{wL^4}{24} \quad ...(iv)$$

Equation (iii) is known as *slope equation* and equation (iv) as *deflection equation*. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of $x = L$ is substituted in these equations.

Let θ_B = Slope at the free end B i.e., $\left(\frac{dy}{dx}\right)$ at B

y_B = Deflection at the free end B.

From equation (iii), we get slope at B as

$$EI.\theta_B = \frac{w}{6}(L-L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\therefore \quad \theta_B = -\frac{wL^3}{6EI} = -\frac{WL^2}{6EI} \quad (\because W = \text{Total load} = w.L) \quad ...(13.5)$$

From equation (iv), we get the deflection at B as

$$EIy_B = -\frac{w}{24}(L-L)^4 - \frac{wL^3}{6} \times L + \frac{wL^4}{24}$$

$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

$$\therefore \quad y_B = -\frac{wL^4}{8EI} = -\frac{WL^3}{8EI} \quad (\because W = w.L)$$

$\therefore$ Downward deflection at B,

$$y_B = \frac{wL^4}{8EI} = \frac{WL^3}{8EI} \quad ...(13.6)$$

PROBLEMS:

3. A cantilever of length 2.5 m carries a uniformly distributed load of 16.4 KN per metre length. If I = 7.95 X 10^7 mm^4 and E = 2 X 10^5 N/mm^2, determine the deflection at the free end.

GIVEN DATA:

L = 2.5 m = 2500 mm

w = 16.4 KN/m, W = w X L = 16.4 X 2.5 = 41000 N

I = 7.95 X 10^7 mm^4

E = 2 X 10^5 N/mm^2

SOLUTION:

1. DEFLECTION AT THE FREE END,

$$y_B = WL^3/8EI = \frac{41000 \times 2500^3}{8 \times 2 \times 10^5 \times 7.95 \times 10^7}$$

= 5.036 mm.

4. A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the deflection at the free end is 40 mm, find the slope at the free end.

GIVEN DATA:

$L = 3\ m = 3000\ mm$

$y_B = 40\ mm$

SOLUTION:

1. SLOPE AT THE FREE END,

$y_B = WL^3/8EI$

$$40 = \frac{WL^2 \times L}{8\ EI} = \frac{WL^2 \times 3000}{8\ EI}$$

$$\frac{WL^2}{EI} = \frac{40 \times 8}{3000}$$

Slope at the free end,

$$\theta_B = WL^2 / 6\ EI = WL^2 / EI \times (1/6)$$

$$= \frac{40 \times 8}{3000} \times (1/6)$$

$$\mathbf{= 0.01777\ rad.}$$

5. A cantilever 120 mm wide and 200 mm deep is 2.5 m long. What is the uniformly distributed load which the beam can carry in order to produce a deflection of 5 mm at the free end? Take E = 200 GN/m^2.

GIVEN DATA:

$L = 2.5\ m = 2500\ mm$

$E = 200\ GN/m^2 = 2 \times 10^5\ N/mm^2$

$b = 120\ mm$

$d = 200\ mm$

$y_B = 5\ mm$

$I = bd^3/12 = 120 \times 200^3 / 12$

$= 8 \times 10^7\ mm^4$

SOLUTION:

1. UDL,

$W = w \times L = 2.5 \times w = 2.5\ w\ N.$

$y = WL^3/8EI$

$$5 = \frac{2.5\ w \times 2500^3}{8 \times 2 \times 10^5 \times 8 \times 10^7}$$

or

$$w = \frac{5 \times 8 \times 2 \times 10^5 \times 8 \times 10^7}{2.5 \times 2500^3} = 16384 \text{ N/m}$$

$= \mathbf{16.384 \text{ kN/m.}}$ **Ans.**

13.5. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig. 13.4.

The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by $AC'B'$ in which portion $C'B'$ is straight.

Let θ_C = Slope at C, *i.e.*, $\left(\frac{dy}{dx}\right)$ at C

y_C = Deflection at point C, and

y_B = Deflection at point B.

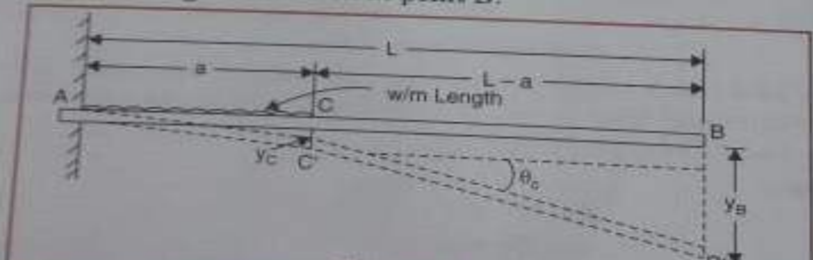


Fig. 13.4

The portion AC of the cantilever may be tak...

The upward deflection of point B due to upward uniformly distributed load acting on the portion AC = upward deflection of C + slope at $C \times CB$

$$= \frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a \qquad (\because CB = a)$$

∴ Net downward deflection of the free end B is given by

$$y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right] \qquad \text{...(13.[illegible])}$$

Problem 13.5. *Determine the slope and deflection of the free end of a cantilever of length 3 m which is carrying a uniformly distributed load of 10 kN/m over a length of 2 m from the fixed end.*

Take $I = 10^8$ mm^4 and $E = 2 \times 10^5$ N/mm^2.

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

U.d.l., $w = 10$ kN/m $= 10000$ N/m $= \frac{10000}{1000}$ N/mm $= 10$ N/mm

Length of u.d.l. from fixed end, $a = 2$ m $= 2000$ mm.

Value of $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let θ_B = Slope of the free end and

y_B = Deflection at the free end.

(*i*) Using equation (13.7), we have

$$\theta_B = \frac{wa^3}{6EI} = \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8} = \mathbf{0.00066.} \quad \textbf{Ans.}$$

(*ii*) Using equation (13.8), we get

$$y_B = \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L-a)$$

$$= \frac{10 \times 2000^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8} \times (3000 - 2000)$$

$= 1 + 0.67 = \mathbf{1.67 \text{ mm.}}$ **Ans.**

Problem 13.6. *A cantilever of length 3 m carries a uniformly distributed load of 10 kN/m over a length of 2 m from the free end. If $I = 10^8$ mm^4 and $E = 2 \times 10^5$ N/mm^2; find : (i) slope at the free end, and (ii) deflection at the free end.*

Sol. Given :

Length, $L = 3$ m $= 3000$ mm

U.d.l., $w = 10$ kN/m $= 10000$ N/m $= \frac{10000}{1000}$ N/mm $= 10$ N/mm

Length of u.d.l. from free end, $a = 2$ m $= 2000$ mm

Value of $I = 10^8$ mm^4

Value of $E = 2 \times 10^5$ N/mm^2

Let θ_B = Slope at the free end *i.e.*, $\left(\frac{dy}{dx}\right)$ at B and

y_B = Deflection at the free end.

(*i*) Using equation (13.9), we get

$$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI}$$

$$= \frac{10 \times 3000^3}{6 \times 2 \times 10^5 \times 10^8} - \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8}$$

$$= 0.00225 - 0.000083 = \mathbf{0.002167\ rad.} \quad \text{Ans.}$$

(*ii*) Using equation (13.10), we get

$$y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$

$$= \frac{10 \times 3000^4}{8 \times 2 \times 10^5 \times 10^8} - \left[\frac{10(3000-2000)^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8} \times 2000\right]$$

$$= 5.0625 - [0.0625 + 0.1667] = \mathbf{4.8333\ mm.} \quad \text{Ans.}$$

Problem 13.7. *A cantilever of length 3 m carries two point loads of 2 kN at the free end and 4 kN at a distance of 1 m from the free end. Find the deflection at the free end. Take $E = 2 \times 10^5$ N/mm² and $I = 10^8$ mm⁴.*

Sol. Given :

Length, L = 3 m = 3000 mm

Load at free end, W_1 = 2 kN = 2000 N

Load at a distance one m from free end,

W_2 = 4 kN = 4000 N

Distance AC, a = 2 m = 2000 mm

Value of $E = 2 \times 10^5$ N/mm²

Value of $I = 10^8$ mm⁴

Let y_1 = Deflection at the free end due to load 2 kN alone

y_2 = Deflection at the free end due to load 4 kN alone.

Fig. 13.6

Downward deflection due to load 2 kN alone at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{2000 \times 3000^3}{3 \times 2 \times 10^5 \times 10^8} = 0.9 \text{ mm.}$$

Downward deflection at the free end due to load 4 kN (*i.e.*, 4000 N) alone at a distance 2 m from fixed end is given by (13.4) as

$$y_2 = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a)$$

$$= \frac{4000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{4000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$$= 0.54 + 0.40 = 0.94 \text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2$$

$$= 0.9 + 0.94 = \mathbf{1.84\ mm.} \quad \text{Ans.}$$

Problem 13.8. *A cantilever of length 2 m carries a uniformly distributed load of 2.5 kN/m run for a length of 1.25 m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and $E = 1 \times 10^4$ N/mm².*

Sol. Given :

Length, L = 2 m = 2000 mm

U.d.l., w = 2.5 kN/m = 2.5 × 1000 N/m

$$= \frac{2.5 \times 1000}{1000} \text{ N/mm} = 2.5 \text{ N/mm}$$

Point load at free end, W = 1 kN = 1000 N

Distance AC, a = 1.25 m = 1250 mm

Width, b = 12 cm

Depth, d = 24 cm

Value of $$I = \frac{bd^3}{12} = \frac{12 \times 24^3}{12}$$

$$= 13824 \text{ cm}^4 = 13824 \times 10^4 \text{ mm}^4 = 1.3824 \times 10^8 \text{ mm}^4$$

Value of $E = 1 \times 10^4$ N/mm²

Let y_1 = Deflection at the free end due to point load 1 kN alone

y_2 = Deflection at the free end due to u.d.l. on length AC.

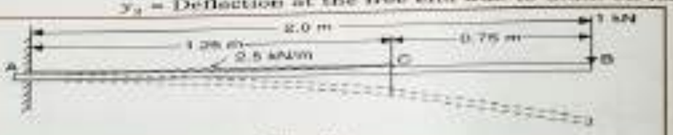


Fig. 13.7

(*i*) Now the downward deflection at the free end due to point load of 1 kN (or 1000 N) at the free end is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 10^4 \times 1.3824 \times 10^8} = 1.929 \text{ mm.}$$

(*ii*) The downward deflection at the free end due to uniformly distributed load of 2.5 N/mm on a length of 1.25 m (or 1250 mm) is given by equation (13.8) as

$$y_2 = \frac{wa^4}{8EI} + \frac{w \cdot a^3}{6EI}(L-a)$$

$$= \frac{2.5 \times 1250^4}{8 \times 10^4 \times 13824 \times 10^8} + \frac{2.5 \times 1250^3}{6 \times 10^4 \times 13824 \times 10^7}(2000 - 1250)$$
$$= 0.5519 + 0.4415 = 0.9934$$

∴ Total deflection at the free end due to point load and u.d.l.

$$= y_1 + y_2 = 1.929 + 0.9934 = \mathbf{2.9224\ mm.} \quad \textbf{Ans.}$$

Problem 13.8. *A cantilever of length 2 m carries a uniformly distributed load 2 kN/m over a length of 1 m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if E = 2.1 × 10⁵ N/mm² and I = 6.667 × 10⁷ mm⁴.*

Sol. Given : (See Fig. 13.8)

Length, $L = 2\text{ m} = 2000\text{ mm}$

U.d.l., $w = 2\text{ kN/m} = \frac{2 \times 1000}{1000}\text{ N/mm} = 2\text{ N/mm}$

Length BC, $a = 1\text{ m} = 1000\text{ mm}$

Point load, $W = 1\text{ kN} = 1000\text{ N}$

Value of $E = 2.1 \times 10^5\text{ N/mm}^2$

Value of $I = 6.667 \times 10^7\text{ mm}^4$

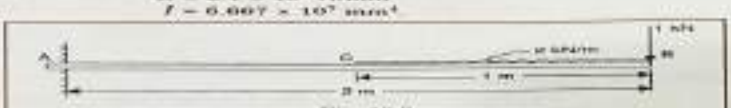


Fig. 13.8

(i) *Slope at the free end*

Let θ_1 = Slope at the free end due to point load of 1 kN *i.e.*, 1000 N

θ_2 = Slope at the free end due to u.d.l. on length BC.

The slope at the free end due to a point load of 1000 N at B is given by equation (13.1 A) as

$$\theta_1 = \frac{WL^2}{2EI} \qquad (\because\ \theta_B = \theta_1 \text{ here})$$
$$= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.0001428\text{ rad.}$$

The slope at the free end due to u.d.l. of 2 kN/m over a length of 1 m from the free end is given by equation (13.9) as

$$\theta_2 = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \qquad (\because\ \theta_B = \theta_2 \text{ here})$$
$$= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000 - 1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$
$$= 0.0001904 - 0.0000238 = 0.0001666\text{ rad.}$$

∴ Total slope at the free end

$$= \theta_1 + \theta_2 = 0.0001428 + 0.0001666 = \mathbf{0.0003094\ rad.} \quad \textbf{Ans.}$$

871

(ii) *Deflection at the free end*

Let y_1 = Deflection at the free end due to point load of 1000 N

y_2 = Deflection at the free end due to u.d.l. on length BC.

The deflection at the free end due to point load of 1000 N is given by equation (13.2 A) as

$$y_1 = \frac{WL^3}{3EI} \qquad (\because\ \text{Here } y_1 = y_B)$$
$$= \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = 0.1904\text{ mm.}$$

The deflection at the free end due to u.d.l. of 2 N/mm over a length of 1 m from the free end is given by equation (13.10) as

$$y_2 = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]$$
$$= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \left[\frac{2(2000-1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} + \frac{2(2000-1000)^3 \times 1000}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}\right]$$
$$= 0.2857 - [0.01785 + 0.0238] = 0.244\text{ mm}$$

∴ Total deflection at the free end

$$= y_1 + y_2 = 0.1904 + 0.244 = \mathbf{0.4344\ mm.} \quad \textbf{Ans.}$$

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QUESTION BANK:

1. What are the methods used for determining slope and deflection?
2. What is the slope and deflection equation for simply supported beam carrying UDL through out the length?
3. What is a Macaulay’s method?
4. What is moment area method?
5. Define : Conjugate beam.
6. Find the slope and deflection of a simply supported beam carrying a point load at the centre using moment area method.
7. Distinguish between actual beam and conjugate beam.
8. A beam 4m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1^o, find the deflection at the centre of the beam.
9. A cantilever of length 2 m carries a point load of 30 KN at the free end and another load of 30 KN at its centre. If EI = 1013 N.mm2 for the cantilever then determine slope and deflection at the free end by moment area method.
10. Determine slope at the left support, deflection under the load and maximum deflection of a simply supported beam of length 10 m, which is carrying a point load of 10 kN at a distance of 6 m from the left end. Take E = 2 x 105 N/mm2 and I = 1 x 108 mm4.
11. A cantilever of length 3 m is carrying a point load of 25 KN at the free end. If I = 108 mm4 and E = 2.1 X 105 N/mm2, then determine slope and deflection of the cantilever using conjugate beam method.
12. A simply supported beam of length 5 m carries a point load of 5 kN at a distance of 3m from the left end. If E = 2 x 105 N/mm2 and I = 108 mm4, determine the slope at the left support and deflection under the point load using conjugate beam method.