What is a Fourier Series?

The concept of vectors can directly be extended to signals due to the analogy between signals and vectors. A Fourier Series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. Fourier series makes use of the orthogonality relationships of the sine and cosine functions.

Since infinite cosine functions and infinite sine functions are mutually orthogonal / exclusive. So it is possible to represent any function as the sum of infinite sine and cosine functions (or linear combination of sine and cosine functions which is known as Fourier series representation.)

It is possible to represent a given signal in Fourier series for one period which implies that the Fourier series is applicable for periodic signals only.

Let x(t) be a periodic signal with fundamental period T then x(t) can be represented in Fourier Series form as -

$x(t) = \sum_{n=0}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$

a_n and b_n are Fourier series coefficients.

Conditions for Existence of Fourier Series

The conditions for existence of Fourier series includes both necessary and sufficient conditions. These are known as Dirichlet's Conditions. These have been given as below

(i) Signal should be absolutely integrable over one period

$\int_T |x(t)|\, dt < \infty$

(ii) Signal must have finite number of maximas and minimas in one period.

(iii) Signal must have finite number of discontinuities in one period.

First one is the **necessary condition** while the remaining two are the **sufficient conditions**

Forms of Fourier Series

Fourier series can be expressed into three different forms. These are been given as follows -

A. Trigonometric Fourier series (TFS)

B. Compact Form / Polar Form Fourier series

C. Exponential Fourier series

A. Trigonometric Fourier Series

A periodic signal x(t) can be represented in the form of trigonometric Fourier series containing sine and cosine terms –

$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots\dots\dots + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t$

$x(t) = \sum_{n=0}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]\ ;\ t_0 \le t \le t_0 + T$

Or

$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))\ ;\ t_0 \le t \le t_0 + T$

Where $\omega_0 = \frac{2\pi}{T}$ and a_0, a_n, b_n are coefficients of Trigonometric Fourier Series. ω_0 is fundamental frequency and $2\omega_0$, $3\omega_0$.... are called the harmonics of ω_0.

a_0 is known as DC term and its value is given by -

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

It is clear from above equation that a_0 is the average value or DC component of x(t) over one period. Now, the coefficients a_n and b_n are being calculated as follows

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_T x(t)\sin(n\omega_0 t)\,dt$$

a_0, a_n and b_n represent the similarity of the signal x(t) associated with DC, cosine and sine function respectively.

Example 1:

Find harmonics and TFS coefficients of the following signals.

1. $x_1(t) = \cos(\pi t + 30$□$+ \sin(2t)$
2. $x_2(t) = 10\ \cos^2(\frac{4}{5}t - 45^0)$
3. $x_3(t) = \cos(7t) - 2\sin(2t + 20°) + 5\cos\left(\frac{4}{7}t + 10°\right) - \sin(0.2t)$
4. $x_4(t) = -4\sin(0.8\pi t) + 2\cos(2\pi t + 30°)$

Solution:

1. $x_1(t) = \cos(\pi t + 30°) + \sin 2t$

$\omega_1 = \pi,\ \omega_2 = 2$

$\frac{\omega_1}{\omega_2} = \frac{\pi}{2} =$ irrational

⇒ $x_1(t)$ is non-periodic

⇒ Fourier series does not exist for $x_1(t)$

2. $x_2(t) = 10\cos^2\left(\frac{4}{5}t - 45°\right) = 10\,\frac{\left(1+\cos\left(\frac{8}{5}t-90°\right)\right)}{2}$

$x_2(t) = 5 + 5\cos\left(\frac{8}{5}t - 90°\right)$

$x_2(t) = 5 + 5\sin\left(\frac{8}{5}t\right)$

TFS –

$x_2(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{8}{5}nt\right) + b_n\sin\left(\frac{8}{5}nt\right)\right)$

$= a_0 + a_1\cos\left(\frac{8}{5}t\right) + a_2\cos\left(\frac{16}{5}t\right) +.. + b_1\sin\left(\frac{8}{5}t\right) b_2\sin\left(\frac{16}{5}t\right) +..$

By comparing, $a_0 = 5$, $a_1 = 5$

3. $x_3(t) = \cos(7t) - 2\sin(2t + 20°) + 5\cos\left(\frac{4}{7}t + 10°\right) - \sin(0.2t)$

$\omega_1 = 7, \omega_2 = 2, \omega_3 = \frac{4}{7}, \omega_4 = 0.2 = \frac{1}{5}$

$\frac{\omega_1}{\omega_2} = \frac{7}{2} = \text{rational}, \frac{\omega_2}{\omega_3} = \frac{2\times 7}{4} = \text{rational}, \frac{\omega_3}{\omega_4} = \frac{4}{7\times 0.2} = \text{rational}$

Since all the ratios are rational therefore, $x_3(t)$ is periodic with period

$\omega_0 = \frac{GCD(7,2,4,1)}{LCM(1,1,7,5)} = \frac{1}{35}$

$x_3(t) = \cos(35 \times 7\omega_0 t) - 2\sin(35 \times 2\omega_0 t + 20°) + 5\cos\left(35 \times \frac{4}{7}\omega_0 t + 10°\right) -$

$\sin(35 \times 0.2\omega_0 t)$

$x_3(t) = \cos(245\omega_0 t) - 2\sin(70\omega_0 t + 20°) + 5\cos(20\omega_0 t + 10°) - \sin(7\omega_0 t)$

$x_3(t) = -\sin(7\omega_0 t) + 5\cos 10°\cos(20\omega_0 t) - 2\sin 20°\cos(70\omega_0 t) + \cos(245\omega_0 t) -$

$5\sin 10°\sin(20\omega_0 t) - 2\cos 20°\sin(70\omega_0 t)$

By comparison with equation of TFS

$a_{20} = 5\cos 10°$; $a_{70} = -2\sin 20°$, $a_{245} = 1$

$b_7 = -1$; $b_{20} = -5\,\text{Sin}10°$; $b_{70} = -2\cos 20°$

4. $x_4(t) = -4\sin(0.8\pi t) + 2\cos(2\pi t + 30°)$

$\omega_1 = 0.8\pi = \frac{4\pi}{5}, \omega_2 = 2\pi$

$\frac{\omega_1}{\omega_2} = \frac{0.8\pi}{2\pi} = 0.4 = \text{rational}$

$\Rightarrow x_4(t)$ is a periodic signal with period ω_0

$\omega_0 = \frac{GCD(4\pi,2\pi)}{LCM(5,1)} = \frac{2\pi}{5}$

$x_4(t) = -4\sin\left(\frac{5}{2\pi} \times 0.8\pi \times \omega_0 t\right) + 2\cos\left(\frac{5}{2\pi} \times 2\pi \times \omega_0 t + 30°\right)$

$= -4\sin(2\omega_0 t) + 2\cos(5\omega_0 t + 30°)$

$= -4\sin(2\omega_0 t) + 2\cos 30° \cos(5\omega_0 t) - 2\sin 30° \sin(5\omega_0 t)$

$= -4\sin(2\omega_0 t) + 2 \times \frac{\sqrt{3}}{2}\cos(5\omega_0 t) - 2 \times \frac{1}{2}\sin(5\omega_0 t)$

$x_4(t) = -4\sin(2\omega_0 t) + \sqrt{3}\cos(5\omega_0 t) - \sin(5\omega_0 t)$

By comparing it with TFS equation

$b_2 = -4,\ a_5 = \sqrt{3}, b_5 = -1$

B. Polar Fourier Series

This is another form of Fourier Series. It is also known as Compact form / Alternate form / Phasor form Fourier series. Polar form is used to find the spectrums.

We know that sinθ = cos(θ - 90°) and cosθ = sin(θ - 90°)

In Polar form, all functions are represented in terms of cosθ. This form is used to find the magnitude and phase of various frequency components.

Trigonometric Fourier series of x(t) was given as -

$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t))$

This can also be written in polar form as below

$x(t) = s_0 + \sum_{n=1}^{\infty} s_n\cos(n\omega_0 t + \theta_n)$

Expanding the above equation, we get

$x(t) = s_0 + \sum_{n=1}^{\infty}(s_n\cos\theta_n\cos(n\omega_0 t) - s_n\sin\theta_n\sin(n\omega_0 t))$

On comparing the trigonometric and polar form, we get

$a_0 = s_0, a_n = s_n\cos\theta_n\,, b_n = -s_n\sin\theta_n$

Or $s_0 = a_0, s_n = \sqrt{a_n^2 + b_n^2}, \theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$

Spectrum of Trigonometric Fourier Series

$x(t) = s_0 + \sum_{n=1}^{\infty} s_n\cos(n\omega_0 t + \theta_n)$

$= s_0 + s_1\cos(\omega_0 t + \theta_1) + s_2\cos(2\omega_0 t + \theta_2) + s_3\cos(3\omega_0 t + \theta_3) + \cdots$

$x(t) = s_0 + s_1\angle\theta_1 \text{and } \omega = \omega_0 + \ s_2\angle\theta_2 \text{and } \omega = 2\omega_0 + \ s_3\angle\theta_3 \text{ and } \omega = 3\omega_0$

Spectrum of TFS is one sided line spectrum or one sided discrete spectrum which is defined at 0, ω_0, $2\omega_0$, $3\omega_0$, ...

(i) Magnitude Spectrum

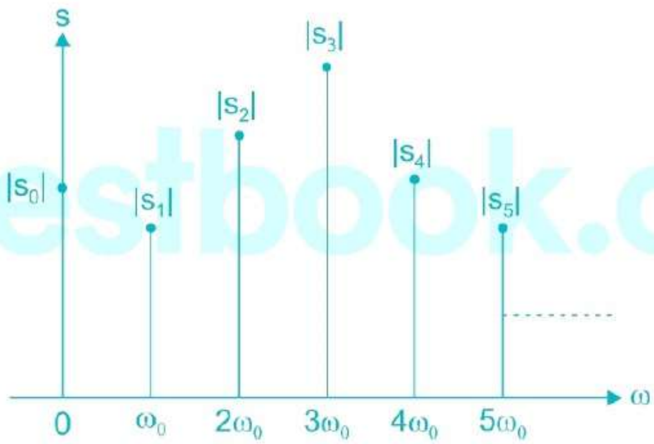
Magnitude spectrum of the signal is being constructed using the following terms

$|s_0|$ → Magnitude associated with DC term. (ω=0)

$|s_1|$ → Magnitude associated with frequency ω_0

$|s_2|$ → Magnitude associated with frequency $2\omega_0$

The magnitude spectrum can be drawn as follows with the values calculated from trigonometric Fourier series coefficients based on the formula given below.



$s_0 = a_0, s_n = \sqrt{a_n^2 + b_n^2}$

(ii) Phase Spectrum

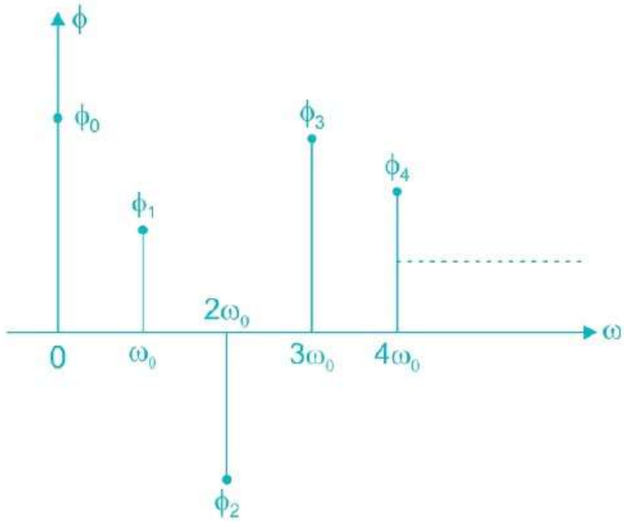
The phase spectrum of the Fourier series consists of the following values

ϕ_0 → Phase associated with DC

ϕ_1 → Phase associated with ω_0

ϕ_n → Phase associated with $n\omega_0$

The phase spectrum is drawn as below with the values calculated from the formula given previously.



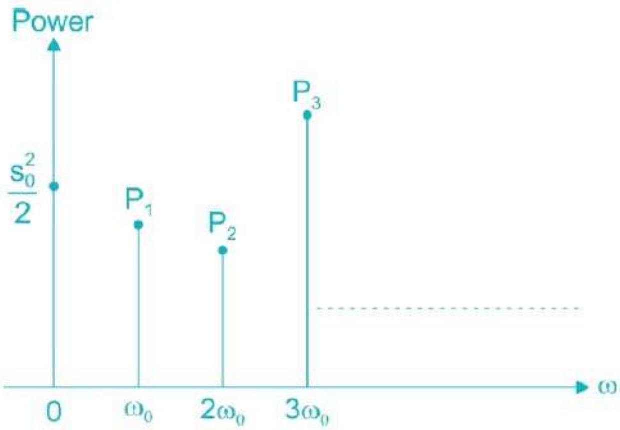
$$\phi_0 = \begin{cases} 0° \,; \text{for} + \text{ve DC} \\ 180° \,; \text{for} - \text{ve DC} \end{cases}$$

$$\phi_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

(iii) Power Spectrum

We have the signal as $x(t) = s_0 + \sum_{n=1}^{\infty} s_n \cos(n\omega_0 t + \theta_n)$.

The power spectrum can be calculated from the formulae given below and can be drawn as following -



$$P_{total} = P_0 + \sum_{n=1}^{\infty} P_n$$

$$P_0 = \frac{s_0^2}{2}, P_n = \frac{s_n^2}{2} = \frac{a_n^2 + b_n^2}{2}; n \neq 0$$

C. Exponential Fourier Series

The exponential Fourier series (EFS) is simpler and more compact. Hence this is most widely used. Trigonometric Fourier series of x(t) is given as follows –

$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$

This can also be written in exponential form as below -

$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{jnw_0t}+e^{-jnw_0t}}{2} + b_n \frac{e^{jnw_0t}-e^{-jnw_0t}}{2j}\right)$

$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n-jb_n}{2}\right)e^{jnw_0t} + \left(\frac{a_n+jb_n}{2}\right)e^{-jnw_0t}$

$= C_0 + \sum_{n=1}^{\infty} C_n e^{jnw_0t} + C_{-n}e^{-jnw_0t}$

$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$ → EFS and C_n are exponential Fourier series coefficients.

Where, $C_0 = a_0$;

$C_n = \frac{a_n-jb_n}{2}$ and $C_{-n} = \frac{a_n+jb_n}{2}$

$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$

Where $C_0 = \frac{1}{T}\int_T x(t)dt$; for $n = 0$

$C_n = \frac{1}{T}\int_T x(t)e^{-jnw_0t}dt$; for $n \neq 0$.

The relation of coefficients of trigonometric and exponential Fourier series are being given as follows-

$c_0 = a_0$	$a_0 = c_0$
$c_n = \frac{a_n-jb_n}{2}$	$a_n = c_n + c_{-n}$
$c_{-n} = \frac{a_n+jb_n}{2}$	$b_n = (jc_n - c_{-n})$

$c_n = c_{-n}^*$

Since c_n and c_{-n} are complex. Hence exponential Fourier series is also known as complex Fourier series (CFS) and coefficients are known as complex coefficients and spectrum is complex spectrum.

Spectrum of Exponential Fourier Series

For EFS, the signal has been represented as follows

$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jnw_0t} = c_0 + c_1e^{jnw_0t} + c_2e^{jnw_0t} + .. + c_{-1}e^{-jw_0t} + c_{-2}e^{-j2w_0t} + ..$

Thus for Exponential Fourier series, spectrum of EFS is two sided line spectrum or two sided discrete spectrum, which is defined at -

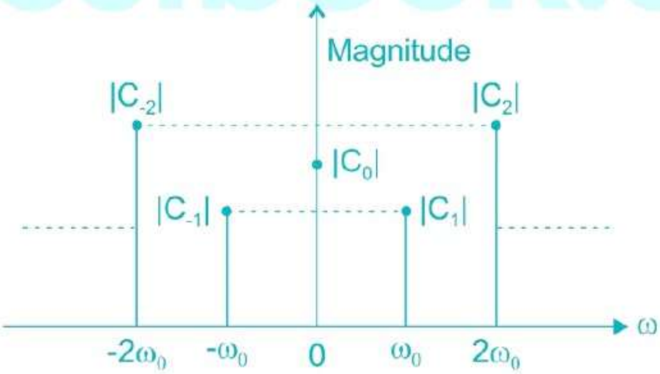
$\omega = 0, \pm\omega_0, \pm2\omega_0, \pm3\omega_0, ...$

There are three components to represent the spectrum. These are magnitude, phase and power spectrum.

(i) Magnitude Spectrum

For magnitude spectrum, we have $|c_0| = |a_0|,\ |c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = |c_{-n}| = \frac{s_n}{2}$

Hence it can be drawn as



Therefore, magnitude spectrum is even function of ω.

(ii) Phase Spectrum

The Phase spectrum for this series is drawn from the following formulae

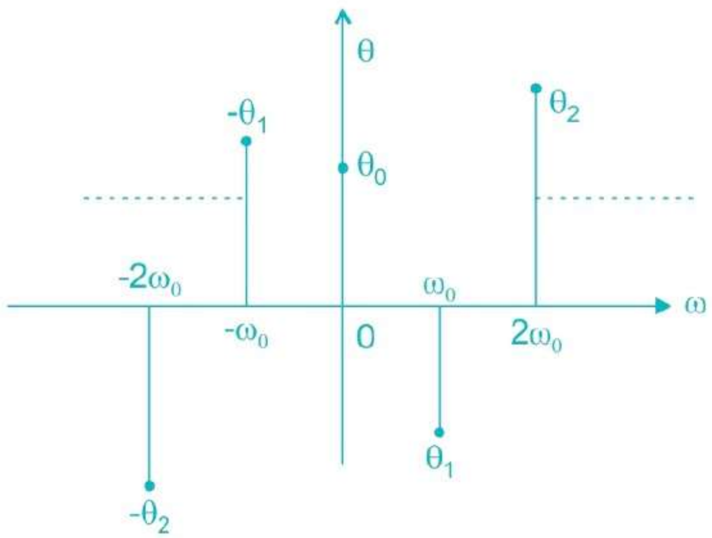
$$\theta_0 = \begin{cases} 0°\ ; \text{for} + \text{ve DC} \\ 180°\ ; \text{for} - \text{ve DC} \end{cases}$$

$$c_n = \frac{a_n - jb_n}{2} \Rightarrow \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$c_{-n} = \frac{a_n + jb_n}{2} \Rightarrow \theta_{-n} = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

$\theta_n = -\theta_{-n}$

A phase spectrum will generally look like the one below



Phase spectrum of EFS is odd function of ω, i.e. anti-symmetric about ω = 0.

(iii) Power Spectrum

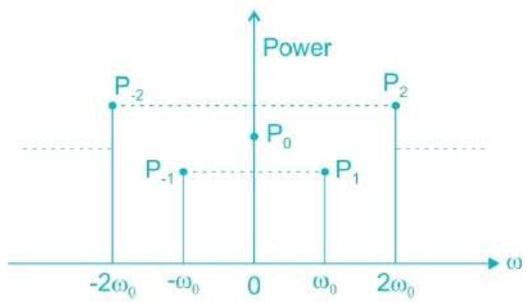
The power spectrum will be drawn through the following formulae

$p_0 = |c_0|^2$

$p_n = |c_n|^2 = \frac{1}{4}(a_n^2 + b_n^2) = \frac{s_n^2}{4}, p_{-n} = |c_{-n}|^2 = \frac{1}{4}(a_n^2 + b_n^2) = \frac{s_n^2}{4}$

$p_n = p_{-n}$

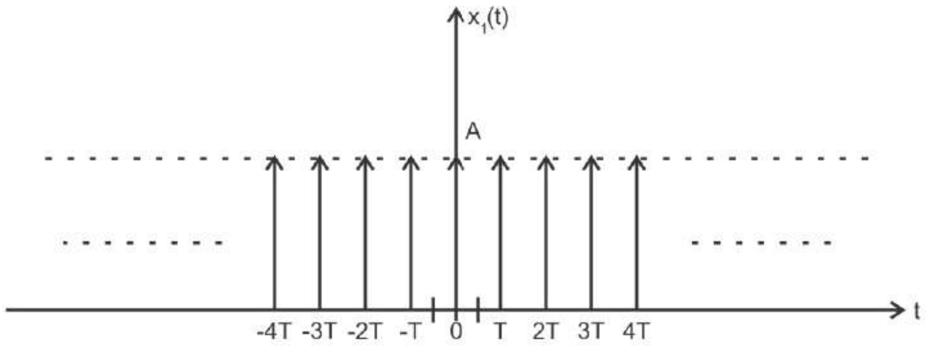
A typical Power Spectrum will be looking like the one below



Power spectrum is even function of ω. i. e. symmetric about $\omega = 0$.

Example 2:

Find the Fourier series of the following signal –



Solution:

$x_1(t)$ represents the periodic train of impulses with period T. Strength A and centered about t=0.

$$x_1(t) = \begin{cases} A\delta(t); & \text{for } t = 0 \\ 0\ ; \text{for } -\frac{T}{2} < t < \frac{T}{2} \text{ and } t \neq 0 \end{cases}$$

and periodic with period $T = \frac{2\pi}{\omega_0}$

$$x_1(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{1}{T}\int_T x_1(t)\,dt \ = \frac{1}{T}\int_{-T/2}^{T/2} A\delta(t) = \frac{A}{T}$$

$$c_n = \frac{1}{T}\int_T x_1(t)e^{-jn\omega_0 t}\ dt = \frac{1}{T}\int_{-T/2}^{T/2} A\delta(t)\, e^{-jn\omega_0 t}dt$$

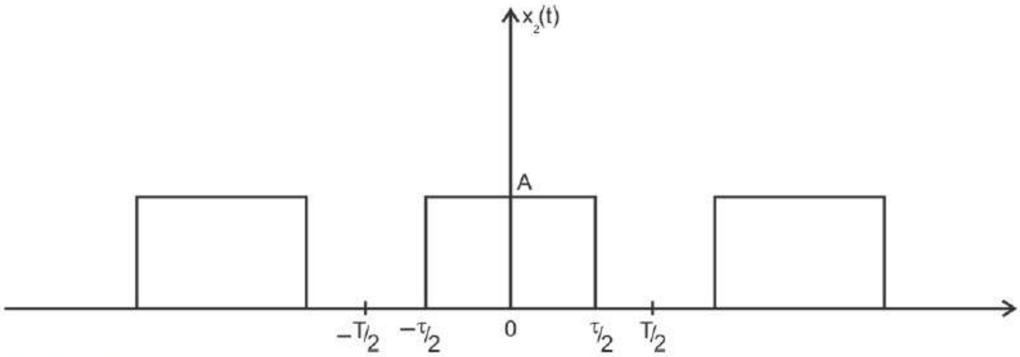
$$= \frac{A}{T}.\, e^{-jn\omega_0 0}\int_{T/2}^{T/2} \delta(t)dt = \frac{A}{T}$$

$$c_n = \frac{A}{T}$$

$$x_1(t) = \frac{A}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 nt} = \frac{A}{T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{T}nt} \rightarrow \text{ Periodic train of impulses}$$

Example 3:

Find the Fourier series of the following signal –



Solution:

$x_2(t)$represents periodic train of pulses with period T, height A, width τ and centered about $t = 0$.

$$x_2(t) = \begin{cases} A\,; \ -{}^{\tau}/_{2} < t < {}^{\tau}/_{2} \\ 0\,; \ -{}^{T}/_{2} < t < -{}^{\tau}/_{2} \text{ and } {}^{\tau}/_{2} < t < {}^{T}/_{2} \end{cases}$$

and periodic with period, $T = \frac{2\pi}{\omega_0}$

$$x_2(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{1}{T}\int_T x_2(t)dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A\tau}{T}$$

$$c_n = \frac{1}{T}\int_T x_2(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\cdot\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-\tau/2}^{\tau/2}$$

$$c_n = \frac{A}{T}\cdot\frac{1}{-jn\omega_0}\left(e^{-jn\omega_0 \tau/2} - e^{jn\omega_0 \tau/2}\right)$$

$$= \frac{A}{T}\frac{2}{-n\omega_0}\frac{e^{-jn\omega_0 \tau/2} - e^{jn\omega_0 \tau/2}}{2j} = \frac{A}{T}\cdot\frac{2}{n\omega_0}\cdot\sin\left(n\omega_0\,{}^{\tau}/_{2}\right)$$

$$= \frac{A}{T}.2\times{}^{\tau}/_{2}\ .\ \frac{\sin\left(n\omega_0\frac{\tau}{2}\right)}{n\omega_0\frac{\tau}{2}} = \frac{A\tau}{T}.\ \frac{\sin\left(n\times\frac{2\pi}{T}\times\frac{\tau}{2}\right)}{n\times\frac{2\pi}{T}\times\frac{\tau}{2}} = \frac{A\tau}{T}.\ \frac{\sin\left(\frac{n\pi\tau}{T}\right)}{\frac{n\pi\tau}{T}}$$

$$c_n = \frac{A\tau}{T}\,.\,\text{sinc}\left(\frac{n\tau}{T}\right)$$

$$x_2(t) = \sum_{n=-\infty}^{\infty}\frac{A\tau}{T}\,.\,\text{sinc}\left(\frac{n\tau}{T}\right).e^{jn\omega_0 t} \quad \rightarrow \quad \text{Periodic train of pulses}$$