Moment Generating Function (MGF – $M_X(t)$)

$$M_X(t) = E[e^{tX}] = \sum_{x=-\infty}^{\infty} e^{tx} p(x) \quad \text{if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \quad \text{if } x \text{ is continuous}$$

Note:

1. $\mu_r' = \left[\frac{d^r}{dt^r} M_X(t) \right]_{t=0}$ is the r^{th} moment from $M_X(t)$.

2. $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

3. r^{th} moment = coefficient of $\frac{t^r}{r!}$

4. If MGF is known, to find mean & variance

$$E(x) = \left[\frac{d}{dt} M_x(t) \right]_{t=0} = M_x'(0).$$

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0} = M_x''(0)$$

mean = $E(x)$

variance = $E(x^2) - [E(x)]^2$

1. If the random variable x has the MGF $M_x(t) = \frac{3}{3-t}$. Find the standard deviation of x

Soln.

$$M_x(t) = \frac{3}{3-t} = 3(3-t)^{-1}$$

$$\frac{d}{dt} M_x(t) = 3(-1)(3-t)^{-2}(-1)$$
$$= 3(3-t)^{-2}$$

$$\frac{d^2}{dt^2} M_x(t) = 3(-2)(3-t)^{-3}(-1)$$
$$= 6(3-t)^{-3}$$

put $t=0$,

$$\left(\frac{d}{dt} M_x(t)\right)_{t=0} = \text{mean}$$

$$\mu_1' = 3(3-0)^{-2} = \frac{3}{9} = \frac{1}{3}$$

$$\mu_2' = \left(\frac{d^2}{dt^2} M_x(t)\right)_{t=0} = 6(3-0)^{-3} = \frac{6}{27}$$

$$\mu_2' = \frac{2}{9}$$

$$\therefore \text{variance} = \mu_2' - (\mu_1')^2$$
$$= \frac{2}{9} - \left(\frac{1}{3}\right)^2$$
$$= \frac{2}{9} - \frac{1}{9}$$

$$\text{Variance} = \frac{1}{9}$$

$$SD = \sqrt{Var(x)} = \sqrt{1/9}$$

$$SD = 1/3$$

2]. A random variable x has the PDF is given by,

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find moment generation function.

Soln.

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx}\, 2e^{-2x}\, dx$$

$$= 2\int_0^{\infty} e^{tx-2x}\, dx$$

$$\cancel{= 2\int_0^{\infty} e^{-(2-t)x}} = 2\int_0^{\infty} e^{-(2-t)x}\, dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$= -\frac{2}{2-t}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-2}{2-t}(0-1)$$

$$M_x(t) = \frac{2}{2-t}$$

3]. Find MGF of a random variable x having PDF

$$f(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Soln.

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-1}^{2} e^{tx} \frac{1}{3}\, dx$$

$$= \frac{1}{3} \int_{-1}^{2} e^{tx}\, dx$$

$$= \frac{1}{3} \left(\frac{e^{tx}}{t}\right)_{-1}^{2}$$

$$M_x(t) = \frac{1}{3t} \left[e^{2t} - e^{-t}\right]$$

4] Find moment generating function of the random variable $x = 1, 2, \ldots$ whose probability mass function $P[X = x] = \frac{1}{2^x}$, $x = 1, 2, \ldots$

Find its mean and variance.

Soln.

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1-\frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2}{2-e^t}\right]$$

$$M_x(t) = \frac{e^t}{2-e^t}$$

mean.:

$$E[x] = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)\,e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} \quad \left\{ d\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2}\right.$$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}$$

$$= \frac{\cancel{2}}{\cancel{2}2} = \frac{2}{(2-1)^2}$$

$$E(x) = 2$$

Variance:

$$Var(X) = E[X^2] - (E[X])^2$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{d}{dt} M_X(t)\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 (2e^t) - 2e^t\, 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \left[\frac{2e^t(2-e^t)^2 + 4e^{2t}(2-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \frac{2(2-1)^2 + 4(2-1)}{(2-1)^4}$$

$$= \frac{2+4}{1}$$

$$E[X^2] = 6$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2$$

$$= 6 - 4$$

$$= 2$$

5) Find MGF of the random variable $x = 1, 2, \ldots$ whose probability $P(X=x) = q^{x-1}p$, $x = 1, 2, \ldots$ find its mean & variance.

Soln.

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x} q^{-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3 + \cdots\right]$$

$$= \frac{p}{q} qe^t \left[1 + qe^t + (qe^t)^2 + \cdots\right]$$

$$= pe^t \left[1 - qe^t\right]^{-1}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

Mean:

$$\frac{d}{dt} M_X(t) = \frac{(1-qe^t)\,pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^{t} - pqe^{2t} + pqe^{2t}}{(1-qe^{t})^{2}}$$

$$\frac{d}{dt} M_X(t) = \frac{pe^{t}}{(1-qe^{t})^{2}}$$

$$\therefore E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \frac{p}{(1-q)^{2}}$$

$$= \frac{p}{p^{2}} \qquad \because p+q=1, \quad p = 1-q$$

$$E(X) = \frac{1}{p}$$

Variance:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\frac{d^2}{dt^2} M_X(t) = \frac{(1-qe^t)^2 pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \frac{(1-q)^2 p - p \times 2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^2 p + 2p^2 q}{p^4} \qquad \because p+q=1,\ p=1-q$$

$$= \frac{p^3 + 2p^2 q}{p^4}$$

$$= \frac{p^2(p+2q)}{p^4}$$

$$E(X^2) = \frac{p+2q}{p^2}$$

$$\therefore Var(X) = \frac{p+2q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{p+2q-1}{p^2}$$

$$= \frac{2q-q}{p^2} \qquad \because p+q=1,\ p-1=-q$$

$$Var(X) = \frac{q}{p^2}$$