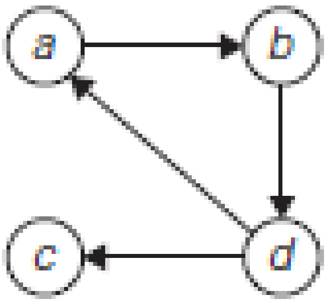
Unit III – Dynamic Programming

- Dynamic Programming
  - Computing a Binomial Coefficient
  - **Warshall's algorithm**
  - Floyd's algorithm
  - Optimal Binary Search Trees
  - Knapsack Problem and Memory functions

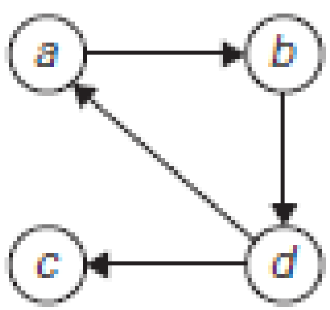
Warshall's algorithm

- Compute the *Transitive closure* of a directed graph
- The ***transitive closure*** of a directed graph with n vertices can be defined as the $n \times n$ boolean matrix $T = \{tij\}$, in which the element in the ith row and the jth column is 1 if there exists a nontrivial path (i.e., directed path of a positive length) from the ith vertex to the jth vertex; otherwise, tij is 0.
- *Example: directed graph (digraph)*



Warshall's algorithm

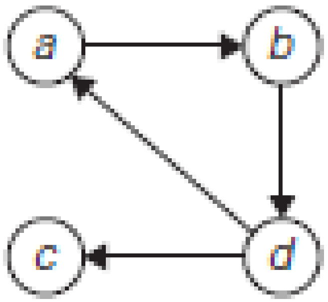
- *Adjacency Matrix* - A= {a_{ij}} of a digraph is the boolean matrix that has 1 in the ith row and jth column if and only if there is a directed edge from i^{th} vertex to j^{th} vertex.



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Warshall's algorithm

- Transitive closure

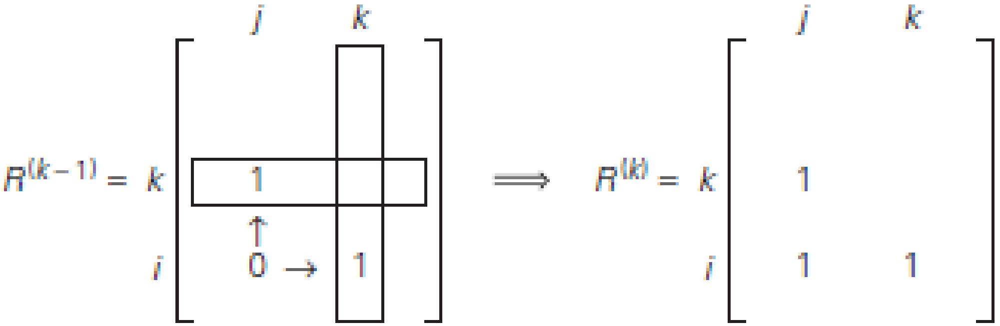


	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	0	0
d	1	1	1	1

Warshall's algorithm

- series of $n \times n$ boolean matrices: $R(0), \ldots, R(k-1), R(k), \ldots R(n)$.
- Matrix value is 1 – using the formula as follows

$$r_{ij}^{(k)} = r_{ij}^{(k-1)} \text{ or } \left(r_{ik}^{(k-1)} \text{ and } r_{kj}^{(k-1)}\right)$$



Rule for changing zeros in Warshall's algorithm.

Warshall's algorithm – Example

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

R_0	a	b	c	D
a	**0**	**1**	**0**	**0**
b	**0**	0	0	1
c	**0**	0	0	0
d	**1**	0	1	0

R_1	a	b	c	D
a	0	**1**	0	0
b	**0**	**0**	**0**	**1**
c	0	**0**	0	0
d	1	**1**	1	0

R_2	a	b	c	D
a	0	1	**0**	1
b	0	0	**0**	1
c	**0**	**0**	**0**	**0**
d	1	1	**1**	1

R_3	a	b	c	D
a	0	1	0	**1**
b	0	0	0	**1**
c	0	0	0	**0**
d	**1**	**1**	**1**	**1**

R_4	a	b	c	D
a	1	1	1	**1**
b	1	1	1	**1**
c	0	0	0	**0**
d	**1**	**1**	**1**	**1**

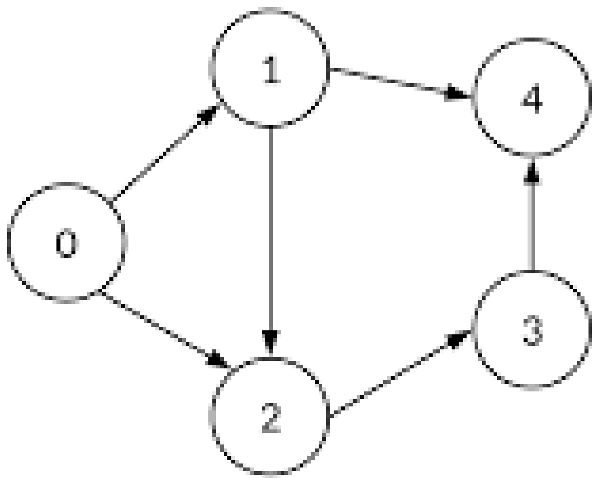
Warshall’s algorithm - Algorithm

ALGORITHM *Warshall*$(A[1..n, 1..n])$

//Implements Warshall’s algorithm for computing the transitive closure
//Input: The adjacency matrix A of a digraph with n vertices
//Output: The transitive closure of the digraph

$R^{(0)} \leftarrow A$
for $k \leftarrow 1$ **to** n **do**
    for $i \leftarrow 1$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            $R^{(k)}[i, j] \leftarrow R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $R^{(k-1)}[k, j])$
return $R^{(n)}$

Warshall's algorithm - Example



Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0