Moments

Discrete R.V.	Continuous R.V.
1st moment: $\mu_1' = E(x) = \sum_{i=1}^{n} x_i P(x_i)$	$\int_{-\infty}^{\infty} x f(x)\, dx$
2nd moment: $\mu_2' = E(x^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$	$\int_{-\infty}^{\infty} x^2 f(x)\, dx$
3rd moment: $\mu_3' = E(x^3) = \sum_{i=1}^{n} x_i^3 P(x_i)$	$\int_{-\infty}^{\infty} x^3 f(x)\, dx$
⋮	
r^{th} moment: $\mu_r' = E(x^r) = \sum_{i=1}^{n} x_i^r P(x_i)$	$\int_{-\infty}^{\infty} x^r f(x)\, dx$

mean : μ_1'

variance : $\mu_2 = \mu_2' - (\mu_1')^2$

I. The density function of Random variable x is $F(x) = Kx(2-x)$, $0 \le x \le 2$. Find K, mean, variance, r^{th} moment.

Soln.

i). Find K.

$$\int_{-\infty}^{\infty} F(x)\, dx = 1$$

$$\int_{0}^{2} Kx(2-x)\, dx = 1$$

$$K \int_{0}^{2} (2x - x^2)\, dx = 1$$

$$K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[\left(4 - \frac{8}{3}\right) - 0\right] = 1$$

$$K\left[\frac{12-8}{3}\right]=1$$

$$K\left[\frac{4}{3}\right]=1$$

$$K=\frac{3}{4}$$

r^{th} moment:

$$\mu_r' = E[x^r]$$

$$= \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^2 x^r \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^{r+1}(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left[2x^{r+1} - x^{r+2}\right]dx$$

$$= \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - 0\right]$$

$$= \frac{3}{4}\,2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{3.\,2^r.\,2^3}{4}\left[\frac{r+3-(r+2)}{(r+2)(r+3)}\right]$$

$$= \frac{3.\,2^r.\,8}{4}\left[\frac{r+3-r-2}{(r+2)(r+3)}\right]$$

$$\mu_r' = E[x^r] = 6.\,2^r\left[\frac{1}{(r+2)(r+3)}\right]$$

mean:

(μ_1')

Put $r=1$ in μ_r'

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\mu_1' = 6.2^1\left[\frac{1}{(1+2)(1+3)}\right]$$

$$= \frac{12}{12}$$

$$\mu_1' = 1$$

mean $= 1 \Rightarrow E(x) = 1$

Variance (μ_2):

$$Var(X) = E(x^2) - [E(x)]^2$$

$$= \mu_2' - \mu_1'^2$$

Now we've to find $E(x^2)$ or μ_2'

put $r=2$ in μ_r',

$$\mu_2' = 6.2^2\left[\frac{1}{(2+2)(2+3)}\right]$$

$$= 6.4\,\frac{1}{4\times 5}$$

$$\mu_2' = \frac{6}{5}$$

$$\therefore Var(x) = \mu_2' - (\mu_1')^2$$

$$= \frac{6}{5} - 1^2$$

$$Var(x) = \frac{1}{5}$$

2]. The continuous random variable X at the period $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find the mean & variance of x.

Soln.

$$\text{mean} = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-1}^{1} x \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2 + x)\, dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} + \frac{1}{2}\right) - \left(\frac{-1}{3} + \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)$$

$$\text{mean} = \frac{1}{3}$$

variance:

$$\text{var}(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)$$

$$E(x^2) = \frac{1}{3}$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9}$$

$$= \frac{3-1}{9}$$

$$Var(x) = \frac{2}{9}$$

HW J. If $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

Find mean and variance.

Results:

* $E(a) = a$
* $E(ax) = a\,E(x)$
* $V(a) = 0$
* $V(ax) = a^2 V(x)$
* $E(ax+b) = a\,E(x) + b$
* $V(ax+b) = a^2 V(x)$
* If x & y are independent, then $E(xy) = E(x) \cdot E(y)$

1. If x and y are independent random variable with variance 2 and 3. Find $var(3x+4y)$

Soln.

Gvn. $var(x) = 2$ and $var(y) = 3$

Now
$$var(3x+4y) = var(3x) + var(4y)$$
$$= 9\,var(x) + 16\,var(y)$$
$$= 9(2) + 16(3)$$
$$= 18 + 48$$
$$var(3x+4y) = 66$$

2. Given the following probability distribution of x.

x	-3	-2	-1	0	1	2	3
$P(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

compute i). $E(x)$ ii). $E(x^2)$ iii). $E(2x \pm 3)$ iv). $var(2x \pm 3)$

Soln.

$$E(x) = \sum_{i=1}^{7} x_i\, P(x_i)$$

$= -3(0.05) - 2(0.1) - 1(0.3) + 0 + 1(0.3) + 2(0.15) + 3(0.1)$

$= 0.25$

ii). $E(x^2) = \sum_{i=1}^{7} x_i^2 P(x_i)$

$= (-3)^2(0.05) + (-2)^2(0.1) + (-1)^2(0.3) + 0 + 1^2(0.3) + 2^2(0.15) + 3^2(0.1)$

$= 2.95$

iii). $E(2x \pm 3) = E(2x) \pm E(3)$

$= 2E(x) \pm 3$

$= 2(0.25) \pm 3$

$= 0.5 \pm 3$

$= 0.5+3,\ 0.5-3$

$= 3.5,\ -2.5$

iv). $var(x) = E(x^2) - (E(x))^2$

$= 2.95 - (0.25)^2$

$= 2.887$

$var(2x \pm 3) = 2^2 \cdot var(x) = 4(2.887)$

$= 11.548$

HW II. If PDF of x is given by $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Show that i). $E(x^r) = \dfrac{2}{(r+1)(r+2)}$

ii). Using the result, to evaluate $E[(2x+1)^2]$