DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT I - FUNDAMENTAL CONCEPTS

Importance of Mathematical Models in Fluid Dynamics

Basic Equations of Fluid Dynamics

The fundamental equations of fluid motion are based on three conservation laws. For most of the engineering applications, the average measurable values of the flow properties are desired, the assumption of continuous distribution of matter is imposed. This assumption is known as **Continuum** and is valid as long as the characteristic length in a physical domain os much larger than the mean free path of molecules.

The basic equations of fluid motion are derived in either integral form or differential form,

1. Conservation of mass – Continuity
2. Conservation of linear momentum – Newton‘s second law
3. Conservation of energy – First law of thermodynamics

The conservation of linear momentum in differential form was derived by Stokes and independently by Navier and therefore is known as the **Navier-Stokes** equation. It is common to refer to the entire system of equations in differential form composed of conservations of mass, momentum, and energy as the **Navier-Stokes equations**. The system of equations may contain nine unknowns they are,

√ Density – ρ

√ Velocity Components u, v, w

√ Total energy - e_t

√ Pressure – p

√ Temperature – T

√ Dynamic viscosity - μ

√ Thermal conductivity – k

There are nine unknowns ad five equations the analytical solution of such a system does not exist. Therefore, numerical techniques are employed to obtain the solution.

Integral Formulations

Integral forms of the equation are used if an average value of the fluid properties at a cross-section is desired. This approach does not provide a detailed analysis of the flow filed, however the application is simple and is used extensively. In general the integral form of the equation is derived for an extensive property and then the conservative laws are applied.

$$\frac{dN}{dt} = \frac{\partial}{\partial t}\int_{C.V.} \eta\rho \, d(\mathrm{Vol}) + \int_{C.S.} \eta(\rho\vec{V}\cdot\vec{n})dS$$ ……Equation 1.1

If N represents an extensive property, then a relation exists between the rate of change of extensive property for a system and the time rate change of the property within the control volume plus the net efflux of the property across the control surfaces. Defining η as the extensive property per unit mass.

Where,

t represents time,

ρ is the density of the fluid

$\vec{V}$ is the velocity vector and

$\vec{n}$ is the unit vector

Conservation of Mass

This conservation law requires that mass is neither created nor destroyed; mathematically this is expressed as dM/dt = 0. In the above equation 1.1 the extensive property N = M and η = 1, the integral form of the conservation of mass is obtained as,

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\ d(\mathrm{Vol}) + \int_{C.S.}\rho\vec{V}\cdot\vec{n}dS = 0$$

……Equation 1.2

The physical interpretation of the above equation is that the sum of the rate of change of mass within the control volume and the net efflux of mass across the control surface is zero.

Conservation of Linear Momentum

Newton's second law applied to a nonaccelerating control volume which is referenced to a fixed coordinate system will result in the integral form of the momentum equation. In the case of linear momentum $G^{\rightarrow} = m\vec{V}^{\rightarrow}$ is taken as extensive property and therefore $\eta = \vec{V}^{\rightarrow}$. Newton's second law in an inertial reference is expressed as $\sum F^{\rightarrow} = dG/dt$. Thus equation 1.1 can be represented as,

$$\Sigma\vec{F} = \frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\ d(\mathrm{Vol}) + \int_{C.S.}\vec{V}(\rho\vec{V}\cdot\vec{n})dS$$

……Equation 1.3

This equation states that the sum of the forces acting on a control volume is equal to the sum of the rate of change of linear momentum within the control volume and the net efflux of the linear momentum across the control surfaces. The forces acting on a control volume usually represent a combination of the body forces and surface forces. For example, x-component of is expressed as,

$$\Sigma F_x = \frac{\partial}{\partial t}\int_{C.V.}\rho u\ d(\mathrm{Vol}) + \int_{C.S.}u(\rho\vec{V}\cdot\vec{n})dS$$

……Equation 1.4

Conservation of Energy

This conservation law is based on the first law of thermodynamics, which is expressed as,

$$\frac{d(\rho e_t)}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial W}{\partial t}$$

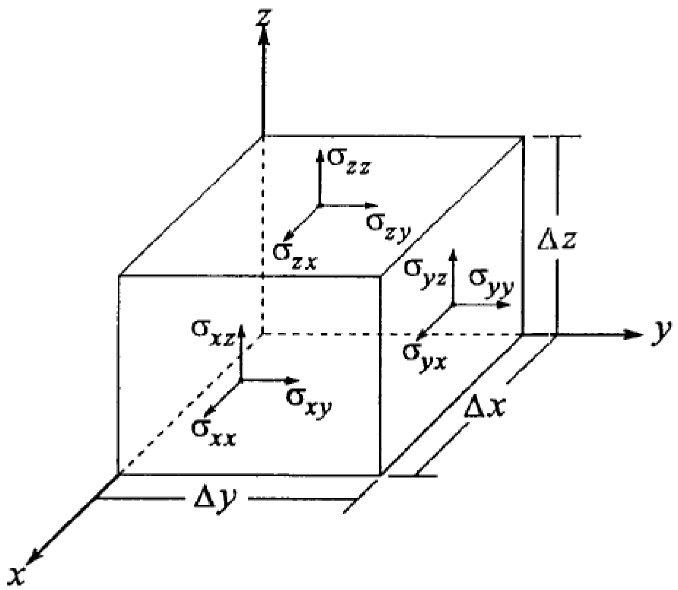
……Equation 1.5 In this relation, e_t represents the total energy of the system per unit mass, while $\partial Q/\partial t$ and $\partial W/\partial t$ represent the rate of heat transfer to the system and the rate of work done on the system is defined as positive. In the above equation 1.1 ρe_t is the extensive property N, and $\eta = e_t$ the total energy per unit mass. Hence,

$$\frac{\partial}{\partial t}\int_{C.V.} \rho e_t \, d(\text{Vol}) + \int_{C.S.} e_t(\rho\vec{V}\cdot\vec{n})dS = \dot{Q} + \dot{W}$$

……Equation 1.6

Differential Formulations

The differential forms of the equations of motion are utilized for situations where a detailed solution of the flow field is required. These equations are obtained by the application of conservation laws to an infinitesimal fixed control volume. A typical differential control volume in a Cartesian coordinate system is shown in below figure 1.2. The differential equations that are derived based on a fixed coordinate system and control volume are known as the Eulerian approach. If the coordinate system and control volume were allowed to move it is called the Lagrangian approach.



Differential element for Cartesian coordinate system

Conservation of Mass

The differential form of the conservation of the mass is known as the Continuity equation, it is written in the vector form as,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

……Equation 1.7

This equation is written in terms of the total derivative as,

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0$$

……Equation 1.8

In the Cartesian coordinate system, the equation can be written as,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

……Equation 1.9

Conservation of Linear Momentum

The linear momentum equation is also known as the Navier-Stokes equation, i.e. obtained by the application of Newton‘s second law to a differential element in an inertial coordinate system. If ζ is used to represent stresses acting on the differential element, the components of the Navier-Stokes equations in a Cartesian coordinate system take the following form;

$$\rho \frac{du}{dt} = \rho f_x + \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})$$

……Equation 1.10

$$\rho \frac{dv}{dt} = \rho f_y + \frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{zy})$$

……Equation 1.11

$$\rho \frac{dw}{dt} = \rho f_z + \frac{\partial}{\partial x}(\sigma_{xz}) + \frac{\partial}{\partial y}(\sigma_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz})$$

……Equation 1.12 Shear stress ζ is usually written in terms of pressure p and viscous stress η. In tensor notation is expressed as,

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

……Equation 1.13

Where δ_{ij} is the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

The components of the linear momentum equation can be written in terms of viscous stresses as,

$$\rho \frac{du}{dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

……Equation 1.14

$$\rho \frac{dv}{dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

……Equation 1.15

$$\rho \frac{dw}{dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

……Equation 1.16

Viscous stresses are related to the rates of strain by a physical law. For most fluids, this relation is linear and is known as Newtonian Fluid. For a Newtonian fluid, viscous stresses in a Cartesian coordinate system are written as,

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \vec{V}$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \vec{V}$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \vec{V}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

Where μ is known as the coefficient of viscosity or dynamic viscosity and λ is defined as the second coefficient of viscosity, the combination of μ and λ in the following form is known as the bulk viscosity k, i.e.

$$k = \lambda + \tfrac{2}{3}\mu$$

If the bulk viscosity of a fluid is assumed negligible, then.

$$\lambda = -\tfrac{2}{3}\mu$$

This is known as Stokes Hypothesis.

Finally, the scalar components of the linear momentum equation in the conservation law form are written as,

X-component of Momentum Equation

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) = \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz})$$

……Equation 1.17

Y-component of Momentum Equation

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) + \frac{\partial}{\partial z}(\rho vw) = \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{yz})$$

……Equation 1.18

Y-component of Momentum Equation

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}(\rho w^2+p)=\frac{\partial}{\partial x}(\tau_{xz})+\frac{\partial}{\partial y}(\tau_{yz})+\frac{\partial}{\partial z}(\tau_{zz})$$

……Equation 1.19

Energy Equation

The energy equation is derived from the first law of thermodynamics it is written as,

$$\frac{\partial}{\partial t}(\rho e_t)+\frac{\partial}{\partial x}(\rho ue_t+pu)+\frac{\partial}{\partial x}(\rho ve_t+pv)+\frac{\partial}{\partial x}(\rho we_t+pw)$$
$$=\frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{xy}+w\tau_{xz}-q_x)+\frac{\partial}{\partial y}(u\tau_{yx}+v\tau_{yy}+w\tau_{yz}-q_y)$$
$$+\frac{\partial}{\partial z}(u\tau_{zx}+v\tau_{zy}+w\tau_{zz}-q_z)$$

……Equation 1.20

These equations can be written in a vector form as

$$\frac{\partial A}{\partial t}+\frac{\partial B}{\partial x}+\frac{\partial C}{\partial y}+\frac{\partial D}{\partial z}=\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}$$

……Equation 1.21

These equations can be written in a vector form as

$$\frac{\partial A}{\partial t}+\frac{\partial B}{\partial x}+\frac{\partial C}{\partial y}+\frac{\partial D}{\partial z}=\frac{\partial E}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial G}{\partial z}$$

……Equation 1.21

Where,

$$A=\begin{Bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho e_t\end{Bmatrix}$$

$$B=\begin{Bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho ue_t+pu\end{Bmatrix}$$

$$C=\begin{Bmatrix}\rho v\\ \rho vu\\ \rho v^2+p\\ \rho vw\\ \rho ve_t+pv\end{Bmatrix}$$

$$D=\begin{Bmatrix}\rho w\\ \rho wu\\ \rho wv\\ \rho w^2+p\\ \rho we_t+pw\end{Bmatrix}$$

$$\boldsymbol{E} = \begin{Bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x \end{Bmatrix}$$

$$\boldsymbol{F} = \begin{Bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - q_y \end{Bmatrix}$$

$$\boldsymbol{G} = \begin{Bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - q_z \end{Bmatrix}$$