DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT I - FUNDAMENTAL CONCEPTS

Incompressible Inviscid Flows – Source, Vortex and Doublet

This section discusses about the numerical analysis of incompressible Inviscid flows. The Incompressible and Inviscid flows are referred as the as the Ideal Flows, i.e. they have ρ = constant and μ = constant. They are flows where the density is constant and the viscosity effects are negligible.

The Uniform Flow

Consider a uniform flow with velocity *V*∞ moving in the *x*-direction, as sketched in Fig. 3.1. This flow is irrotational, and a solution of Laplace's equation for uniform flow yields:

$\boxed{\phi = V_\infty x}$……Equation 1.22

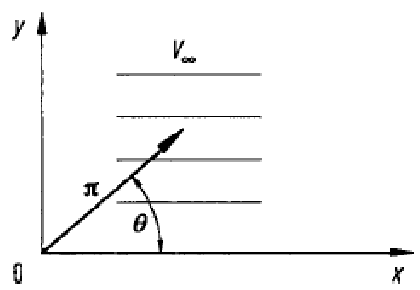

Figure 1.3 The uniform flow

In polar coordinates, (*r*, θ), the above equation can be expressed as

$$\boxed{\phi = V_\infty r \cos\theta}$$

……Equation 1.23

The Source Flow

Consider a flow with straight streamlines emanating from a point, where the velocity along each streamline varies inversely with distance from the point, as shown in Figure 1.4. Such flow is called *source flow*. This flow is also irrotational, and a solution of Laplace's equation yields

$$\phi = \frac{\Lambda}{2\pi} \ln r$$

……Equation 1.24

where Λ is defined as the *source strength*; Λ is physically the rate of volume flow from the source, per unit depth perpendicular to the page in Figure 1.4. If Λ is negative, S*ink flow*, which is the opposite of source flow. In Figure 1.4. point 0 is the origin of the radial streamlines. We can visualize that point 0 is a *point source or sink* that *induces* the radial flow about it; in this interpretation, the point source or sink is a *singularity* in the flow field. We can also visualize that point 0 in Figure 1.4., is simply one point formed by the intersection of the plane of the paper and a *line* perpendicular to the paper. The line perpendicular to the paper is a *line source*, with strength Λ per unit length.

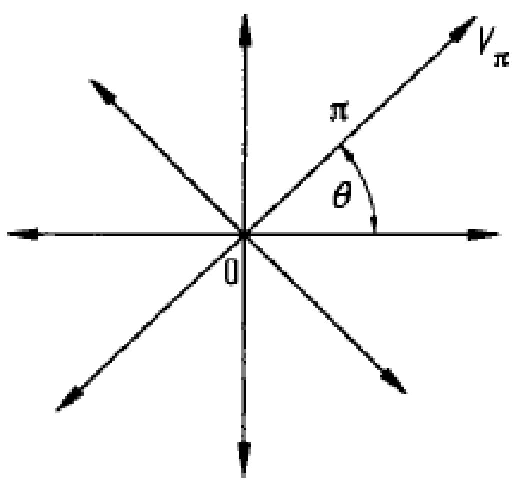

Figure 1.4 Source flow

The Vortex Flow

Consider a flow where all the streamlines are concentric circles about a given point, where the velocity along each streamline is inversely proportional to the distance from the centre, as shown in Figure 1.5. Such flow is called *vortex flow*. This flow is irrotational, and a solution of Laplace's equation yields

$$\phi = -\frac{\Gamma}{2\pi}\theta$$

……Equation 1.25

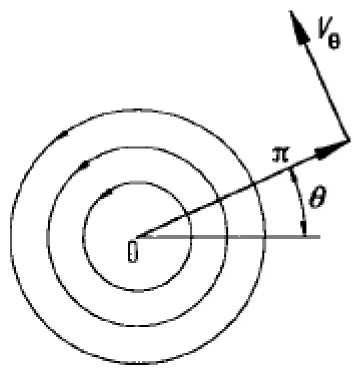

Figure 1.5 Vortex flow

Where Γ is the strength of the vortex. In Figure 1.5., point 0 can be visualized as a *point vortex* that *induces* the circular flow about it; in this interpretation, the point vortex is a *singularity* in the flow field. Visualize that point 0 in Figure 1.5. is simply one point formed by the intersection of the plane of the paper and a line perpendicular to the paper. This line is called a vortex *filament*, of strength Γ. The strength Γ is the *circulation* around the vortex filament, where circulation is defined as

$$\Gamma = -\oint \vec{V} \cdot \vec{\mathrm{d}}s$$

……Equation 1.26

In the above, the line integral of the velocity component tangent to a curve of elemental length d*s* is taken around a closed curve. This is the general definition of circulation. For a vortex filament, the above expression for Γ is defined as the vortex strength.

The Doublet

The source and sink pair leading to a singularity is called a Doublet Flow. The potential at some point P, caused by a doublet at Q, is given by

$$\phi(P) = \mu(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left(\frac{1}{2\pi}\ln\left(r(P,Q)\right)\right)$$

……Equation 1.27

Here μ(Q) is the strength of the doublet and nQ is the direction of the doublet.

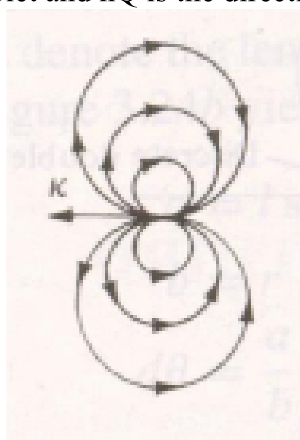

Figure 1.6 Doublet

Once more, we can put a lot of doublets in a row. We then get a doublet distribution. To find the velocity potential at P, we now have to use

$$\phi(P) = \int_S \mu(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left(\frac{1}{2\pi}\ln\left(r(P,Q)\right)\right) ds.$$

……Equation 1.28

1.4 Panel Methods – Lifting Flow over Arbitrary Bodies

Panel method is a technique of approximating the flow by replacing the flow surface by a series of Line segments (2D) or Panels (3D) and placing the distribution of source or vortices or doublets on each panel. The advantages of this method include,

1. No need to define a throughout the flow field
2. Flexibility, i.e. capable of treating wide range of geometries
3. Economy – provides result with in a relative short time

Non-lifting Flows over Arbitrary Bodies – The Source Panel Method is used because source has zero circulation, therefore it is used only for non-lifting cases.

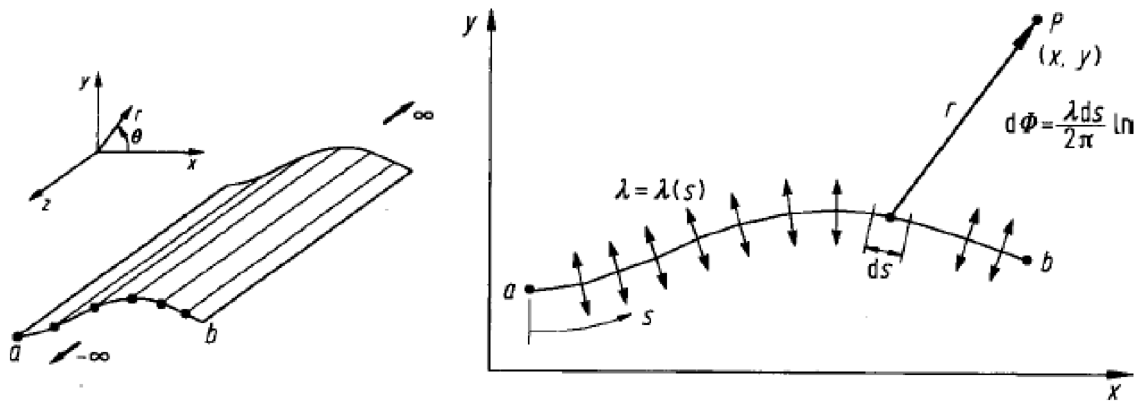

Figure 1.7 The Source Sheet

Lifting Flow over Arbitrary Bodies – The Vortex Panel Method is used because the vortices have circulation and they are used for lifting cases.

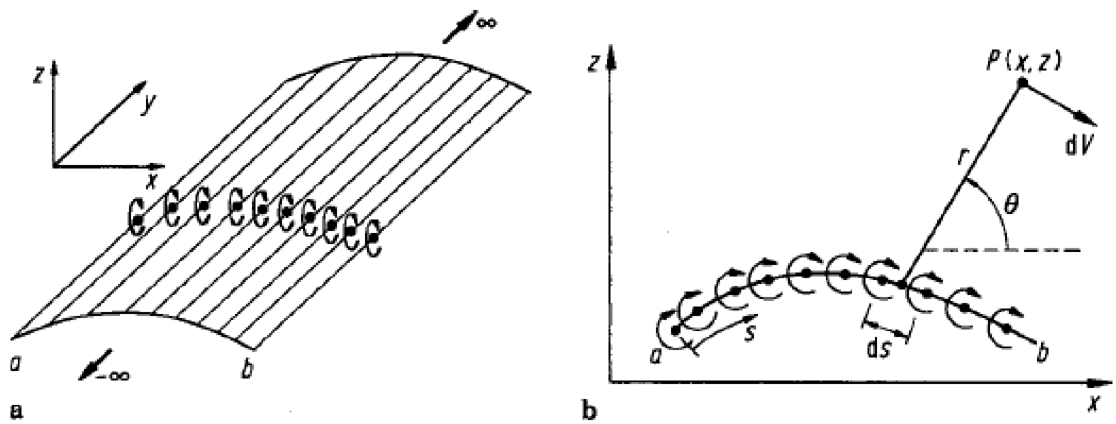

Figure 1.8 The Panel Sheet

In the present section, we introduce the analogous concept of a *vortex sheet*. Consider the straight vortex filament as shown in the above figure 1.8. Now imagine an infinite number of straight vortex filaments side by side, where the strength of each filament is infinitesimally small. These side-by-side vortex filaments form a *vortex sheet*, as shown in perspective in the figure 1.8. If we look along the series of vortex filaments the vortex sheet will appear as sketched at the lower right of Fig. 3.10. Here, we are looking at an edge view of the sheet; the vortex filaments are all perpendicular to the page. Let *s* be the distance measured along the vortex sheet in the edge view. Define $\gamma = \gamma(s)$ as the strength of the vortex sheet, per unit length along *s*. Thus, the strength of an infinitesimal portion d*s* of the sheet is γ d*s*. This small section of the vortex sheet can be treated as a distinct vortex of strength γ d*s*. Now consider point *P* in the flow, located a distance *r* from d*s*. The small section of the vortex sheet of strength γ d*s* induces a velocity potential at *P*, obtained from Equation 1.25 as

$$\mathrm{d}\Phi = -\frac{\gamma \,\mathrm{d}s}{2\pi}\theta$$

……Equation 1.29

The velocity potential at *P* due to the entire vortex sheet from *a* to *b* is

$$\Phi = -\frac{1}{2\pi}\int_a^b \theta\gamma \,\mathrm{d}s$$

……Equation 1.30

In addition, the circulation around the vortex sheet in Fig. 3.10 is the sum of the strengths of the elemental vortices, i.e.

$$\Gamma = \int_a^b \gamma \,\mathrm{d}s$$

……Equation 1.31

Another property of a vortex sheet is that the component of flow velocity tangential to the sheet experiences a discontinuous change across the sheet, given by

$$\gamma = u_1 - u_2$$

……Equation 1.32

where *u*1 and *u*2 are the tangential velocities just above and below the sheet respectively. Equation 1.32 is used to demonstrate that, for flow over an airfoil, the value of γ is zero at the trailing edge of the airfoil. This condition, namely

$$\gamma_{\mathrm{TE}} = 0$$

……Equation 1.33

is one form of the *Kutta condition* which fixes the precise value of the circulation around an airfoil with a sharp trailing edge. Finally we note that the circulation around the sheet is related to the lift force on the sheet through the Kutta–Joukowski theorem**:**

$$L = \rho_\infty V_\infty \Gamma$$

……Equation 1.34

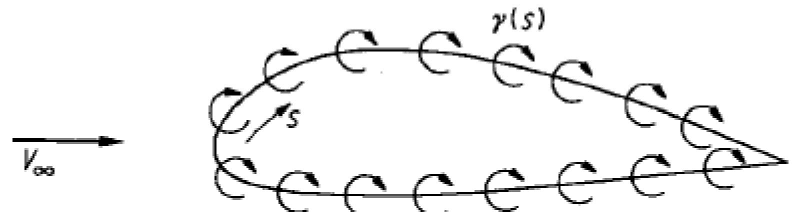

Figure 1.9 Simulation of an arbitrary airfoil by distributing a vortex sheet

Clearly, a finite value of circulation is required for the existence of lift. In the present section, we will see that the ultimate goal of the vortex panel method applied to a given body is to calculate the amount of circulation, and hence obtain the lift on the body from Equation 1.34. With the above in mind, consider an arbitrary two-dimensional body, shown in Figure 1.9. Let us wrap a vortex sheet over the complete surface of the body, as shown in Figure 1.9.We wish to find $\gamma(s)$ such that the body surface becomes a streamline of the flow. This is the purpose of the vortex panel method.

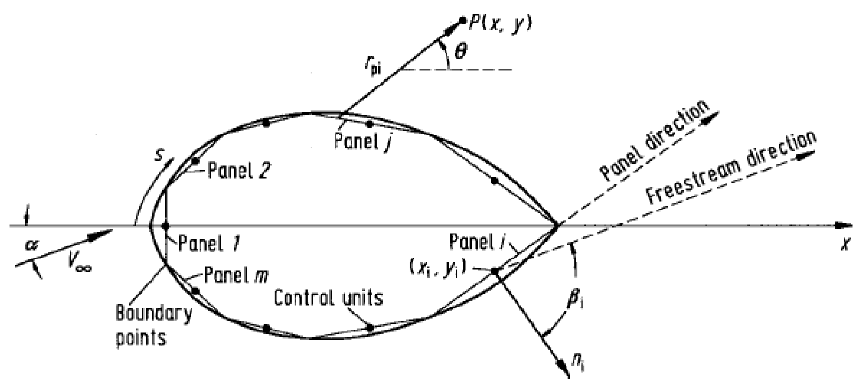

Figure 1.10 Source panel distribution over the surface of a body of arbitrary shape

Let us approximate the vortex sheet shown in Figure 1.9 by a series of straight panels. Let the vortex strength $\gamma(s)$ per unit length be constant over a given panel, but allow it to vary from one panel to the next. That is, for the n panels shown in Figure 1.10, the vortex panel strengths per unit length are $\gamma_1, \gamma_2, \ldots, \gamma_j, \ldots, \gamma_n$. These panel strengths are unknowns; the main thrust of the panel technique is to solve for $\gamma_{j,\,j} = 1$ to n, such that the body surface becomes a streamline of the flow and such that the Kutta condition is satisfied.

Let P be a point located at (x, y) in the flow, and let r_{pj} be the distance from any point on the j^{th} panel to P, as shown in Figure 1.10. The radius r_{pj} makes the angle θ_{pj} with respect to the x-axis. The velocity potential induced at P due to the j^{th} panel, $\Delta\varphi j$, is, from Equation 1.29,

$$\Delta\phi_j = -\frac{1}{2\pi}\int_j \theta_{pj}\gamma_j \, ds_j$$

……Equation 1.35

In Equation 1.35, γ_j is constant over the j^{th} panel, and the integral is taken over the j^{th} panel only. The angle θ_{pj} is given by

$$\theta_{pj} = \tan^{-1}\frac{y - y_j}{x - x_j}$$

……Equation 1.36

In turn, the potential at *P* due to *all* the panels is Equation 1.35 summed over all the panels:

$$\phi(P) = \sum_{j=1}^{n} \phi_j = -\sum_{j=1}^{n}\frac{\gamma_j}{2\pi}\int_j \theta_{pj} \, ds_j$$

……Equation 1.37

Since point *P* is just an arbitrary point in the flow, let us put *P* at the control point of the i^{th} panel shown in Figure 1.10. The coordinates of this control point are (x_i, y_i). Then Equation 1.36 and 1.37 become

$$\theta_{i,j} = \tan^{-1}\frac{y_i - y_j}{x_i - x_j}$$

$$\phi(x_i, y_i) = -\sum_{j=1}^{n}\frac{\gamma_j}{2\pi}\int_j \theta_{ij} \, ds_j$$

……Equation 1.38

Equation 1.38 is physically the contribution of *all* the panels to the potential at the control point of the i^{th} panel. At the control points, the normal component of the velocity is zero; this velocity is the superposition of the uniform flow velocity and the velocity induced by all the vortex panels. The component of V_∞ normal to the i^{th} panel is given

$$V_{\infty,n} = V_\infty \cos\beta_i$$

……Equation 1.39

The normal component of velocity induced at (*x*i, *y*i) by the vortex panels is

$$V_n = \frac{\partial}{\partial n_i}[\phi(x_i, y_i)]$$

……Equation 1.40

Combining Equation 1.38 and 1.40, we have

$$V_n = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} \mathrm{d}s_j$$

……Equation 1.41

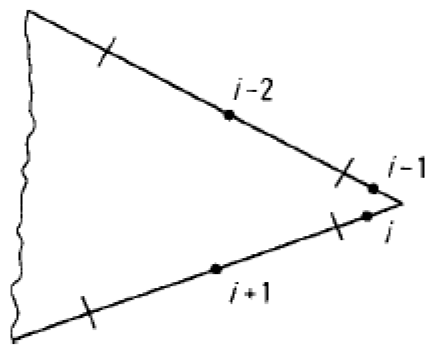

Figure 1.11 Vortex panel at the trailing edge

Equation 1.41 is a linear algebraic equation with *n* unknowns, $\gamma_1, \gamma_2, \ldots, \gamma_n$. It represents the flow boundary condition evaluated at the control point of the i^{th} panel. If Equation 1.41 is applied to the control points of *all* the panels, we obtain a system of *n* linear equations with *n* unknowns. To this point, we have been deliberately paralleling the discussion of the source panel method however, the similarity stops here. For the source panel method, the *n* equations for the *n* unknown source strengths are routinely solved, giving the flow over a non-lifting body. In contrast, for the lifting case with vortex panels, in addition to the *n* equations given by Equation 1.41 applied at all the panels, we must also satisfy the Kutta condition, Equation 1.41. This can be done in several ways. For example, consider Figure 1.11, which illustrates a detail of the vortex panel distribution at the trailing edge. Note that the length of each panel can be different; their length and distribution over the body is up to your discretion. Let the two panels at the trailing edge (panels *i* and $i - 1$ in Figure 1.11) be very small. The Kutta condition is applied *precisely* at the trailing edge and is given by $\gamma_{(TE)} = 0$. To approximate this numerically, if points *i* and $i-1$ are close enough to the trailing edge, we can write

$$\gamma_{\mathrm{i}} = -\gamma_{\mathrm{i=1}}$$

……Equation 1.42

such that the strengths of the two vortex panels *i* and $i - 1$ exactly cancel at the point where they touch at the trailing edge. Thus, in order to impose the Kutta condition on the solution of the flow, Equation 1.42 must be included. Note that Equation 1.41 evaluated at all the panels and Equation 1.42 constitutes an *over-determined* system of *n* unknowns with $n + 1$ equations. Therefore, to obtain a determined system, Equation 1.41 is not evaluated at one of the control points on the body. That is, we choose to ignore one of the control points, and we evaluate Equation 1.41 at the other $n - 1$ control points. This, in combination with Equation 1.42, now gives a system of *n* linear algebraic equations with *n* unknowns, which can be solved by standard techniques.

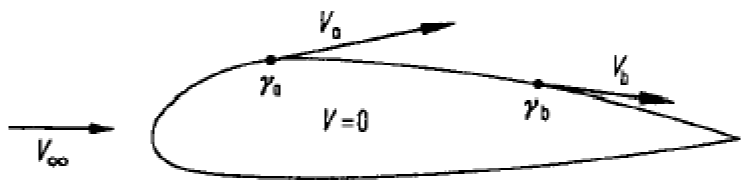

Figure 1.12 Airfoil as a solid body with zero velocity inside the profile

At this stage, we have conceptually obtained the values of $\gamma_1, \gamma_2, \ldots, \gamma_n$ which make the body surface a streamline of the flow and which also satisfy the Kutta condition. In turn, the flow velocity tangent to the surface can be obtained directly from γ. To see this more clearly, consider the airfoil shown in Figure 1.12. We are concerned only with the flow outside the airfoil and on its surface. Therefore, let the velocity be zero at every point *inside* the body, as shown in Figure 1.12. In particular, the velocity just inside the vortex sheet on the surface is zero.

$$\gamma = u_1 - u_2 = u_1 - 0 = u_1$$

……Equation 1.43

u denotes the velocity tangential to the vortex sheet. In terms of the picture shown in Figure 1.12, we obtain $V_a = \gamma_a$ at point *a*, $V_b = \gamma_b$ at point *b*, etc. Therefore, *the local velocities tangential to the airfoil surface are equal to the local values of* γ. In turn, the local pressure distribution can be obtained from Bernoulli's equation.

The total circulation and the resulting lift are obtained as follows. Let s_j be the length of the j^{th} panel. Then the circulation due to the j^{th} panel is $\gamma_j\, s_j$. In turn, the total circulation due to all the panels is

$$\Gamma = \sum_{j=1}^{n} \gamma_j s_j$$

……Equation 1.44

Hence, the lift per unit span is obtained from

$$L' = \rho_\infty V_\infty \sum_{n=1}^{n} \gamma_j s_j$$

……Equation 1.45

1.5 Mathematical Properties of Fluid Dynamic Equations – Elliptical, Parabolic and Hyperbolic Equations

The solution procedure of a partial differential equation (PDE) depends upon the type of equation, thus it is important to study the various classifications of PDE's. Imposition of the initial or boundary condition also depends upon the type of PDE.

Linear and Nonlinear PDE's

Linear PDE: In a Linear PDE the dependent variable and its derivative enter the equation linearly, i.e. there is no product of the dependent variable or its derivatives.

Example: One dimensional wave equation

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = -\boldsymbol{a}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$$

……Equation 1.46

Where, a is the speed of sound which is assumed constant

Nonlinear PDE: A Nonlinear PDE contains product of the dependent variable and its derivative.

Example: Inviscid Burgers equation

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}} = -\boldsymbol{u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$$

……Equation 1.47

Second-Order PDE's

To classify the second-order PDE, Consider the following equation

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G = 0$$

……Equation 1.48

Where the coefficients A, B, C, D, E, F and G are functions of the independent variables x and y and of dependent variable Φ. By the definition we can express $d\Phi_x$ and $d\Phi_y$ as

$$d\phi_x = \frac{\partial\phi_x}{\partial x}dx + \frac{\partial\phi_x}{\partial y}dy = \frac{\partial^2\phi}{\partial x^2}dx + \frac{\partial^2\phi}{\partial x\partial y}dy$$

$$d\phi_y = \frac{\partial\phi_y}{\partial x}dx + \frac{\partial\phi_y}{\partial y}dy = \frac{\partial^2\phi}{\partial x\partial y}dx + \frac{\partial^2\phi}{\partial y^2}dy$$

……Equation 1.49

The equation 1.48 can be expressed as

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = H$$

……Equation 1.50

Where,

$$H = -\left(D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi + G\right)$$

Equations 1.49 and 1.50 are solved for Φ, using the cramers rule we get,

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\begin{vmatrix} A & H & C \\ dx & d\phi_x & 0 \\ 0 & d\phi_y & dy \end{vmatrix}}{\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}}$$

Since it is possible to have the discontinuous in the second order derivatives of the dependent variable across the characteristics, these derivatives are indeterminate. Thus setting the denominator equal to zero.

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

Yields the equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

……Equation 1.51

Solving this quadratic equation yields the equation of the characteristic in physical space.

$$\left(\frac{dy}{dx}\right)_{\alpha,\beta} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

……Equation 1.52

Depending on the value of B^2-4AC the characteristic curves are real or imaginary. They are classified as,

(a) elliptic if $B^2 - 4AC < 0$

(b) parabolic if $B^2 - 4AC = 0$

(c) hyperbolic if $B^2 - 4AC > 0$

Elliptical Equations

A partial differential equation is elliptical in a region if B^2-4AC is less than zero at all points in the region. An elliptic PDE has no real characteristic curves. A disturbance is propagated instantly in all directions within the region. The domain of solution of a elliptical equation is a closed region.

Example:

Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

……Equation 1.53

Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y)$$

……Equation 1.54

Parabolic Equations

A partial differential equation is parabolic in a region if B^2-4AC is equal to zero at all points in the region. A parabolic PDE has solution domain as open region. An parabolic PDE has one real characteristic curve.

Example:

Unsteady heat conduction in one dimension

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

……Equation 1.55

Diffusion of viscosity equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

……Equation 1.56

Hyperbolic Equations

A partial differential equation is called hyperbolic if B^2-4AC is greater than zero at all points in the region. A hyperbolic PDE has two real characteristic curves.

Example:

Second-order wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2}$$

……Equation 1.57