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DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT II – DISCRETIZATION

Stability Properties of Explicit and Implicit Methods

Since the outcome of the competition between explicit and implicit methods is governed by their respective stability properties, a closer look must be given to this issue. First, we observe that the space discretization of a time-dependent partial differential equation produces a system of ordinary differential equations. Consider for example the diffusion equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

After discretization in space using central finite differences, we obtain the following system of ordinary differential equations.

$$\frac{du_i}{dt} = \frac{1}{\Delta x^2}(u_{i-1} - 2u_i + u_{i+1}) \qquad \text{or} \qquad \frac{dU}{dt} = AU + F(t)$$

With

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} \qquad A = \frac{1}{\Delta x^2}\begin{pmatrix} \text{B.C.} & & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & & \text{B.C.} \end{pmatrix}$$

Therefore, the analysis of the stability of a time-stepping scheme for solving the PDE reduces to the analysis of the stability of a time-stepping scheme for solving a system of ODEs. Furthermore, when we consider a periodic solution in space, i.e. $u = u(t)e^{ik_m x}$, the system of ODEs reduces to a single ODE. Indeed, inserting the periodic solution hypothesis and realizing that similarly, we obtain $u_{i-1} = u_i e^{-ik_m \Delta x}$

$$\frac{du_i}{dt} = \frac{e^{ik_m\Delta x} - 2 + e^{-ik_m\Delta x}}{\Delta x^2} u_i = \underbrace{-\frac{4}{\Delta x^2}\sin^2\left(\frac{k_m\Delta x}{2}\right)}_{q} u_i$$

i.e. an ODE whose coefficient q depends on the reduced wavenumber $km\Delta x$, the locus of q (in the complex plane) being called the Fourier footprint of the discretized equation. The stability analysis can then be reduced to the stability analysis for a single ordinary differential equation $du/dt = qu$, where q is a complex coefficient.

Definition:

The stability of the numerical integration of an ordinary differential equation is usually defined by the following statement. A method is said to be stable (weakly stable) if the numerical solution remains bounded when the number of steps n goes to infinity and the time step size Δt goes to zero with the product $n\Delta t$ remaining constant.

Stability Properties:

Weak Instability

The results of the calculation are displayed in Fig. 1. One notices that the perturbation on $u1$ gives rise to amplifying oscillations. In fact, as small as the initial perturbation may be - and there will always be one because of round-off errors – it will eventually lead to an explosion of the numerical solution. This phenomenon is unacceptable. It is named *weak instability*.

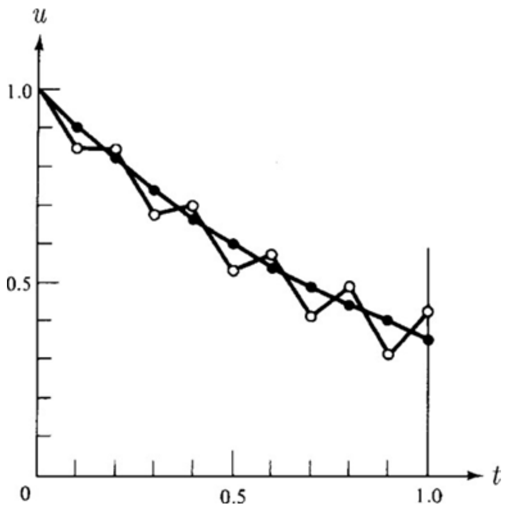


Fig. 1 Numerical solution of $du\ dt = -u$; $u(o) = 1$ with the 2-step explicit midpoint method

Region of (absolute) stability

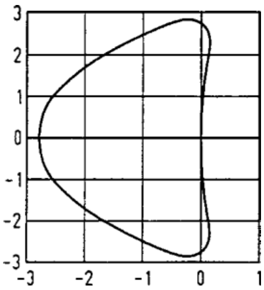
The concept of region of (absolute) stability was introduced by Dahlquist. The region of (absolute) stability of a numerical algorithm for integrating an O.D.E. is the set of values of $z = q\Delta t$ (q = complex parameter of the test equation $du/dt = qu$) such that the sequence un of numerical values remains bounded as $n\to\infty$. As the definition of stability requires that the sequence un remain bounded for $n\to\infty$, $\Delta t \to 0$, this is equivalent to stating that the origin lies in the region of (absolute) stability [$\Delta t\to 0$ implies $z = q\Delta t\to 0$].

Stiff Problems

Problems, where there is such a coexistence of phenomena with very disparate time scales, are called stiff problems. Unfortunately, they are not uncommon in many fields of engineering in particular in fluid mechanics. For those problems, it would be desirable to have at our disposal schemes such that a physically stable problem would lead to a bounded solution irrespective of the value of the time step Δt. That property is called absolute stability or A – A-stability.

Absolute Stability

Absolute stability was defined as a property by which the numerical solution of a physically stable problem would be bounded, irrespective of the time step. Let us translate this in mathematical terms. Test problems of the type $du/dt = qu$ are stable if $Re(q) \leq 0$. Therefore, the set of values of $q\Delta t$, corresponding to stable problems is the left half plane. Absolute stability is thus equivalent to requiring that the region of stability include the left-half plane.



Region of stability of the Runge-Kutta method

Absolute stability ≡ the region of stability includes the left-half plane.