DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT II – DISCRETIZATION

Conservative Upwind Discretization for Hyperbolic Systems

It will be shown how conservative upwind discretizations of hyperbolic systems such as the Euler equations can be constructed.

When systems of conservation laws like the Euler equations are considered, the extension of upwind schemes poses a problem, in the sense that wave speeds of both signs can be simultaneously present. Indeed, the characteristic speeds associated with the unsteady 1D Euler equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = 0$$

Are u, $u+a$, and $u-a$, so that speeds of both signs exist when the flow is subsonic? It is then impossible to use a biased discretization of the whole flux vector **F** since this would lead to a downwind discretization for one of the waves. If one considers the quasi-linear form of the equations, then one can decompose the original system in characteristic equations and upwind each equation according to the corresponding wave speed sign (Courant-Isaacson-Rees scheme [8]) but this approach does not satisfy the conservation property which is crucial for the correct treatment of discontinuities (Part I, Sect. 2.9). This is the main reason why schemes based on a central space discretization such as the Lax-Wendroff scheme and schemes involving artificial diffusion have been so popular in the sixties and seventies. Indeed, these schemes are indifferent to wave speed signs and therefore extend readily to systems:

artificial diffusion $$h_{i+1/2} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - d(\mathbf{U}_{i+1} - \mathbf{U}_i)$$

Lax-Wendroff $$h_{i+1/2} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - \frac{\mathbf{A}_{i+1/2}\Delta t}{2\Delta x}(\mathbf{F}_{i+1} - \mathbf{F}_i)$$

The early eighties have seen the development of conservative upwind schemes, which have since become extremely popular, because of their crisp resolution of discontinuities and their superior ability in following moving shock waves. The remainder of this section will therefore be devoted to a brief presentation of the two major families of conservative upwind schemes.

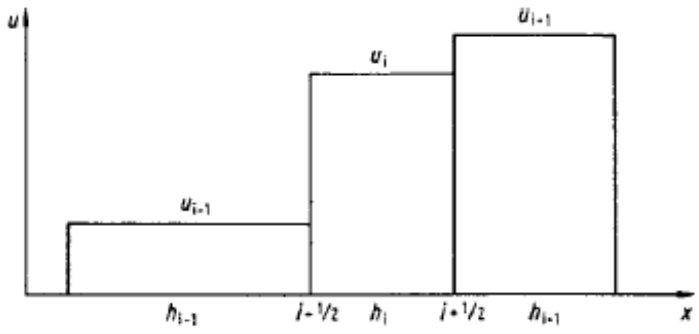
Flux Difference Splitting (FDS) Schemes — Approximate Riemann Solvers

The starting point of Flux Difference Splitting scheme is the scheme developed in the late fifties by the Russian mathematician Godunov [17] for the unsteady 1D Euler equations. This scheme is based on the integral form of the equations.9 The integral form of the unsteady 1D Euler equations (9.36) is

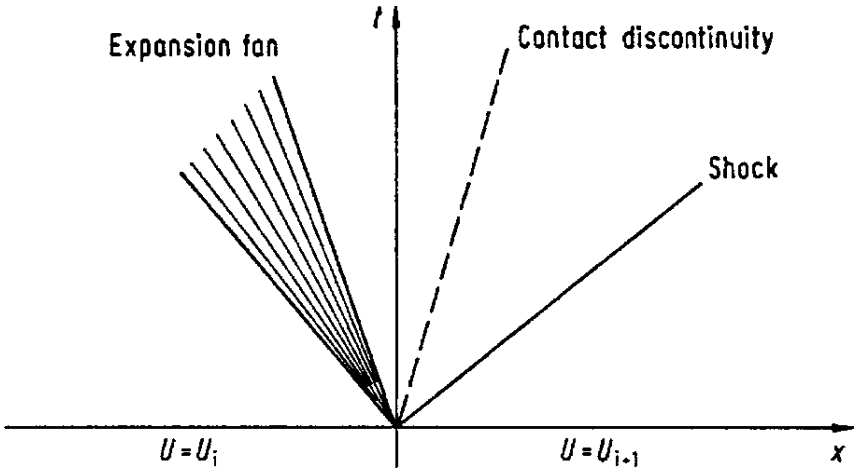
$$\frac{d}{dt}\int_a^b \mathbf{U}dx + \mathbf{F}(\mathbf{U}_b) - \mathbf{F}(\mathbf{U}_a) = 0$$

For the numerical solution of the problem, the domain of interest is divided up into intervals (cells in the finite volume terminology) and the unknowns of the numerical solution **U***i* are the *average* flow quantities over the corresponding interval (see Fig. 9.13) rather than point values as in the finite difference method. The boundaries of interval *i* are noted *i*±1/2. As illustrated in the figure, the intervals need not be of constant length (*hi*−1 _ *hi* _ *hi*+1). The first step in Godunov's method consists in *reconstructing* a piecewise continuous distribution of the flow variables from the cell averages. The simplest choice is a piecewise constant reconstruction as illustrated in the figure.10 At the interval interfaces, the flow variable distributions are thus discontinuous. Now, there exists an *exact* solution of the 1D Euler equations for initial data consisting of two constant states separated by a discontinuity—this problem is known in the literature as the Riemann problem, and applies in particular to the flow in a shock tube. The solution consists of elementary waves (shock wave, contact discontinuity, expansion wave) originating from the interface, as illustrated in Fig. 9.14 for the shock tube problem. An interesting property of the solution is that flow variables are constant along straight lines in *x*−*t* space (which implies that the solution is self-similar). In particular, it is constant in time at the location of the interface. As long as the two solutions at each interface of an interval do not interact (which imposes an upper bound on the time step), it is thus possible to compute the *exact*

solution at the new time level from the initial piecewise constant data. This constitutes the second step in Godunov's method, called the *evolution* step. From



Finite volume representation



Schematic representation of the solution of the Riemann problem

the exact solution at the new time level, it is then possible to compute the new cell averages in order to restart the process. This constitutes the third step of the method, called *projection* step.

Actually, it is possible to compute directly the cell averages at the new time level without computing the details of the solution. Indeed, integrating in time between *tn* and *tn*+1 = *tn*+Δ*t* the integral form of the equations applied to interval *i*, one finds

$$\int_{i-1/2}^{i+1/2} \mathbf{U}^{n+1} dx - \int_{i-1/2}^{i+1/2} \mathbf{U}^{n} dx + \int_{t^n}^{t^{n+1}} \mathbf{F}_{i+1/2} dt - \int_{t^n}^{t^{n+1}} \mathbf{F}_{i-1/2} dt = 0$$

This expression simplifies greatly since $\mathbf{U}^n = \mathbf{U}_i^n$ = const over the interval and $\mathbf{F}_{i\pm1/2}$ are constant over the time step. In addition, $\mathbf{U}_i^{n+1}$ being the average over the interval of the solution at the new time level, $\int_{i-1/2}^{i+1/2} \mathbf{U}^{n+1} dx = \mathbf{U}_i^{n+1} h_i$. The final result is thus

$$(\mathbf{U}_i^{n+1} - \mathbf{U}_i^n)h_i + (\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2})\Delta t = 0$$

or dividing through by $hi\Delta t$,

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}}{h_i} = 0$$

from which we deduce that Godunov's scheme is a conservative discretization of the 1D Euler equations with the numerical flux function

$$h_{i+1/2} = \mathbf{F}(\mathbf{U}_{\text{exact}}(x_{i+1/2}, t))$$

combined with forward Euler time stepping. That this is an upwind discretization clearly shows up by applying it to the linear advection equation. Since the exact solution of the linear advection equation is the initial solution moving with speed c, it results that (for $c > 0$)

$$h_{i+1/2} = c\,u_i \qquad \text{and} \qquad h_{i-1/2} = c\,u_{i-1}$$

and one recovers the first-order upwind discretization.

The essential drawback of Godunov's scheme is that the computation of **U** exact (xi+1/2, t) requires the solution of a non-linear algebraic problem, i.e. it is computationally expensive. Now, as most of the information contained in the exact solution is lost by the averaging process, Roe [34] suggested to replace the exact Riemann problem by a linearized problem

$$\frac{\partial \mathbf{U}}{\partial t} + \tilde{\mathbf{A}}_{i+1/2} \frac{\partial \mathbf{U}}{\partial x} = 0$$

where $\tilde{\mathbf{A}}_{i+1/2}$ is a function of $\mathbf{U}_i$ and $\mathbf{U}_{i+1}$, chosen to satisfy certain properties:

1. $\tilde{\mathbf{A}}(\mathbf{U}, \mathbf{U}) = \mathbf{A}(\mathbf{U})$;
2. $\tilde{\mathbf{A}}_{i+1/2}$ has a complete set of real eigenvalues for any pair of $\mathbf{U}_i$, $\mathbf{U}_{i+1}$;
3. $\tilde{\mathbf{A}}_{i+1/2}(\mathbf{U}_{i+1} - \mathbf{U}_i) = \mathbf{F}_{i+1} - \mathbf{F}_i$.

The first condition is required for consistency, the second ensures that the linearized problem has a solution, and the third condition is a sufficient condition for the scheme to be conservative. It also has the nice additional property that the solution of the linearized problem is identical to the solution of the exact problem when a single wave is present.

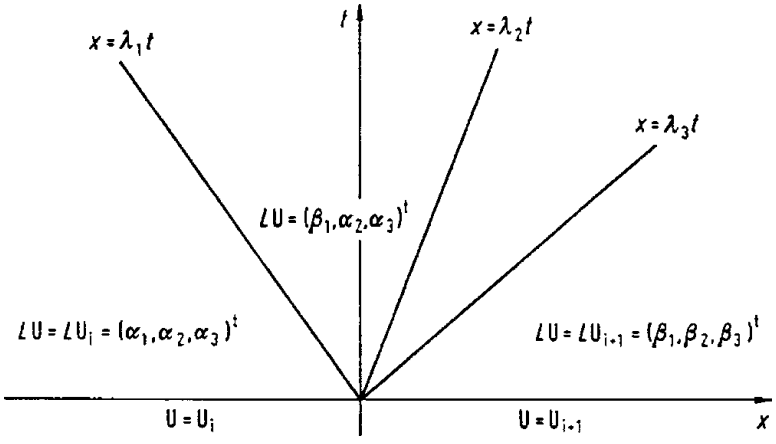
Now, the solution of the linearized problem is found quite easily by the theory of characteristics. Multiplying the linearized equation by the matrix L of left eigenvectors of $\tilde{\mathbf{A}}_{i+1/2}$, one obtains

$$L\frac{\partial \mathbf{U}}{\partial t} + L\tilde{\mathbf{A}}_{i+1/2}\frac{\partial \mathbf{U}}{\partial x} = L\frac{\partial \mathbf{U}}{\partial t} + \Lambda L\frac{\partial U}{\partial x} = 0$$

Where Λ is the (diagonal) matrix of eigenvalues of $\tilde{\mathbf{A}}_{i+1/2}$. These are decoupled linear advection equations for the characteristic variables, components of the vector $L\mathbf{U}$. For the 1D Euler equations, there are three components. Noting

$$L\mathbf{U}_i = (\alpha_1, \alpha_2, \alpha_3)^t; \qquad L\mathbf{U}_{i+1} = (\beta_1, \beta_2, \beta_3)^t$$

and arranging the eigenvalues in increasing order, the solution of the linear problem is schematically shown in Fig. 9.15 (in terms of characteristic variables) and for the case of the figure ($\lambda 1 < 0, \lambda 2, \lambda 3 > 0$),



Solution of the linearized Riemann problem

$$L\mathbf{U}_{i+1/2} = (\beta_1, \alpha_2, \alpha_3)^t \qquad \rightarrow \mathbf{U}_{i+1/2} = R(\beta_1, \alpha_2, \alpha_3)^t$$

where R is the matrix of right eigenvectors of $\tilde{\mathbf{A}}_{i+1/2}$ (inverse of L). The corresponding flux is thus (the flux being linear)

$$\begin{aligned}\mathbf{F}_{i+1/2} &= \mathbf{F}_i + \tilde{\mathbf{A}}_{i+1/2}(\mathbf{U}_{i+1/2} - \mathbf{U}_i) = \mathbf{F}_i + \tilde{\mathbf{A}}_{i+1/2} R(\beta_1 - \alpha_1, 0, 0)^t \\ &= \mathbf{F}_i + R\Lambda(\beta_1 - \alpha_1, 0, 0)^t = \mathbf{F}_i + R\Lambda^- L(\mathbf{U}_{i+1} - \mathbf{U}_i) \\ &= \mathbf{F}_i + \tilde{\mathbf{A}}^-_{i+1/2}(\mathbf{U}_{i+1} - \mathbf{U}_i)\end{aligned}$$

or

$$\begin{aligned}\mathbf{F}_{i+1/2} &= \mathbf{F}_{i+1} - \tilde{\mathbf{A}}_{i+1/2}(\mathbf{U}_{i+1} - \mathbf{U}_{i+1/2}) = \mathbf{F}_{i+1} - \tilde{\mathbf{A}}_{i+1/2} R(0, \beta_2 - \alpha_2, \beta_3 - \alpha_3)^t \\ &= \mathbf{F}_{i+1} - R\Lambda(0, \beta_2 - \alpha_2, \beta_3 - \alpha_3)^t = \mathbf{F}_{i+1} - R\Lambda^+ L(\mathbf{U}_{i+1} - \mathbf{U}_i) \\ &= \mathbf{F}_{i+1} - \tilde{\mathbf{A}}^+_{i+1/2}(\mathbf{U}_{i+1} - \mathbf{U}_i)\end{aligned}$$

relations from which it appears that the flux difference $\mathbf{F}i+1 - \mathbf{F}i$ has been split into a positive and a negative part to calculate $\mathbf{F}i+1/2$, whence the name Flux Difference Splitting. By this splitting of the flux difference, the scheme automatically adapts the difference scheme to the local flow quantities. It is thus a solution-adaptive differencing scheme as alluded to in the introduction.

Averaging the two previous expressions, the following (third) form of Roe's scheme is obtained:

$$\mathbf{F}_{i+1/2} = \frac{\mathbf{F}_i + \mathbf{F}_{i+1}}{2} - \frac{1}{2}|\tilde{\mathbf{A}}_{i+1/2}|(\mathbf{U}_{i+1} - \mathbf{U}_i)$$

$$\text{where } |\tilde{\mathbf{A}}_{i+1/2}| = \tilde{\mathbf{A}}^+_{i+1/2} - \tilde{\mathbf{A}}^-_{i+1/2}.$$

Now, this has exactly the same form as the artificial diffusion flux formula (9.37) except that the diffusion coefficient is replaced here by a diffusion *matrix*.

The Flux Difference Splitting approach pioneered by Roe has met with a considerable success. Several schemes of this type, also called Approximate Riemann solvers, were developed since the beginning of the eighties [13, 15, 33, 38], among which the most popular is certainly Osher's scheme [33].

Flux Vector Splitting (FVS) Schemes

The idea of flux vector splitting was introduced in computational fluid dynamics by Steger and Warming [40]. The idea had been previously proposed in astrophysics by Sanders and Prendergast [36] but was rediscovered independently by Steger and Warming. The starting point of Steger and Warming's scheme is the observation that the compressible inviscid fluxes are homogeneous functions of degree 1 in the conservative variables. Consequently, by a theorem due to Euler,

$$\mathbf{F}(\mathbf{U}) = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}\mathbf{U} = \mathbf{A}(\mathbf{U})\mathbf{U}$$

Now, the flux Jacobian matrix A is fully diagonalizable and it is possible to split it between its positive and negative parts (see previous paragraph)

$$\mathbf{A} = R\Lambda L = R(\Lambda^+ + \Lambda^-)L = \underbrace{R\Lambda^+ L}_{\mathbf{A}^+} + \underbrace{R\Lambda^- L}_{\mathbf{A}^-}$$

to which correspond the split fluxes

$$\mathbf{F}^+ = \mathbf{A}^+\mathbf{U} \qquad \mathbf{F}^- = \mathbf{A}^-\mathbf{U}$$

Now, the split fluxes **F**± being associated with positive (respectively negative) eigenvalues only, it is possible to use upwind difference formulas to discretize the corresponding flux derivatives.

The Steger and Warming flux vector splitting suffers from a lack of continuity at those points where an eigenvalue of **A** vanishes (stagnation and sonic points).

To remedy this problem, van Leer developed an alternative, continuous, flux vector splitting [45], which is no longer based on the homogeneity property of the inviscid flux vectors. The basic requirements are

- ✓ the split fluxes sum up to the whole flux: **F**+ +**F**−= **F**;
- ✓ the split fluxes Jacobians have positive (respectively negative) eigenvalues only;
- ✓ **F**⁻= 0 for supersonic flow (respectively **F**+ = 0 for supersonic flow with negative velocity).

Van Leer imposed a few additional requirements in particular to ensure the crisp resolution of discontinuities.

The flux vector splitting approach and van Leer's scheme in particular have become extremely popular in the CFD community [37, 43], but it was soon realized that flux vector splitting schemes are excessively dissipative at contact discontinuities (boundary and shear layers) [46]. To avoid this while keeping the robustness of flux vector splitting schemes, an improved flux vector splitting scheme was recently developed by Liou and Steffen [30](AUSM scheme). Jameson's CUSP scheme [25], although formulated in the artificial diffusion formalism, appears essentially equivalent to this latter scheme. Finally, Coquel and Liou [6] have proposed a procedure to construct hybrid flux vector/flux difference splitting schemes to combine the robustness of the flux vector splitting schemes with respect to strong shock and expansion waves and the accuracy of flux difference splitting schemes with respect to contact discontinuities. They examine in particular the van Leer/Osher hybrid, which provides results of comparable accuracy as Osher's scheme for viscous flow calculations at a cost only slighly superior to van Leer's FVS scheme.