DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT II – DISCRETIZATION

Advantages of Upwind Differencing

Let us illustrate the effects of numerical dissipation by considering a couple of examples. Let us suppose we are looking for a numerical solution to the advection equation given below

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Let us first select the space discretization scheme. We consider two possibilities: the central finite difference and the backward (upwind) finite difference formulas.

BACKWARD (UPWIND)

$$\frac{du_i}{dt} + c\frac{u_i - u_{i-1}}{\Delta x} = 0$$

CENTRAL

$$\frac{du_i}{dt} + c\frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$

To select the time-integration scheme, let us first calculate the Fourier footprints of the discretized equations. Introducing the periodic solution hypothesis

$(u = U(t)e^{Ikx} \rightarrow u^n_{i\pm 1} = u^n_i e^{\pm I\eta})$,

We get

$$\frac{du_i}{dt} = \underbrace{-c\frac{1-e^{-I\eta}}{\Delta x}}_{q} u_i$$

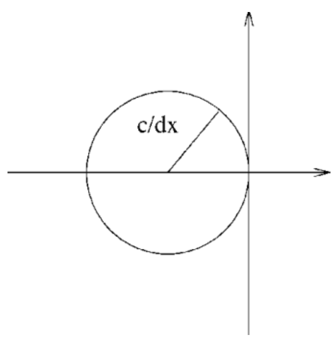
$$\begin{aligned}\frac{du_i}{dt} &= -c\frac{e^{I\eta}-e^{-I\eta}}{2\Delta x}u_i \\ &= \underbrace{-I\frac{c\sin\eta}{\Delta x}}_{q} u_i\end{aligned}$$

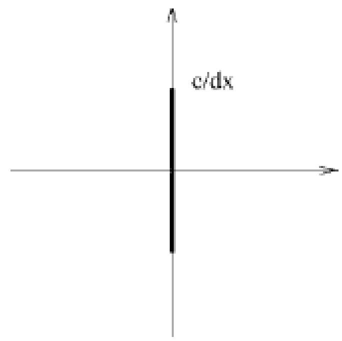
The discretized equation reduce to the model equation

$$\frac{du}{dt} = qu \qquad u(0) = 1$$

Where the q coefficient depends on the reduced wave number η .

The locus of q is the Fourier footprint of the discretized equation. They are shown below for the two discretization schemes





Now, the time-integration scheme should be selected so that $q\Delta t$ can lie within the region of stability. By comparing the respective loci of q with the region of stabilities of some of the schemes examined previously ,it appears quite clearly that the forward Euler scheme cannot be used together with the central space discretization, but the mid-point method, on the contrary, can, and the opposite conclusion applies to the backward finite difference discretization. Applying the mid-point method and the forward Euler method to the central and backward space discretization's respectively, the fully discrete schemes and their truncation errors are respectively

FIRST-ORDER UPWIND-FORWARD EULER

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + c\frac{u_i-u_{i-1}}{\Delta x} = 0$$

$$\text{TE} = O(\Delta x, \Delta t)$$

LEAPFROG (CENTRAL SPACE-MID POINT)

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} + c\frac{u_{i+1}-u_{i-1}}{2\Delta x} = 0$$

$$\text{TE} = O(\Delta x^2, \Delta t^2)$$

and, being central in both space and time, the leapfrog method is seen to be of superior accuracy. Let us now compute the amplification factors of both schemes.

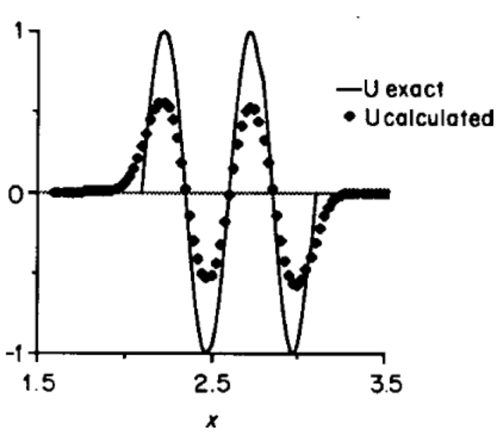
$$g = 1 + q\Delta t \quad \rightarrow$$

$$\begin{aligned} g &= 1 - \frac{c\Delta t}{\Delta x}(1 - e^{-I\eta}) \\ &= 1 - \nu(1 - e^{-I\eta}) \\ &= 1 - \nu(1 - \cos\eta) - I\nu\sin\eta \\ |g|^2 &= 1 - 2\nu(1-\cos\eta) + \\ &\qquad \nu^2[(1-\cos\eta)^2 + \sin^2\eta] \\ &= 1 - 2\nu(1-\cos\eta) + 2\nu^2(1-\cos\eta) \\ &= 1 - 2\nu(1-\nu)(1-\cos\eta) < 1 \text{ for } 0 < \nu < 1 \end{aligned}$$

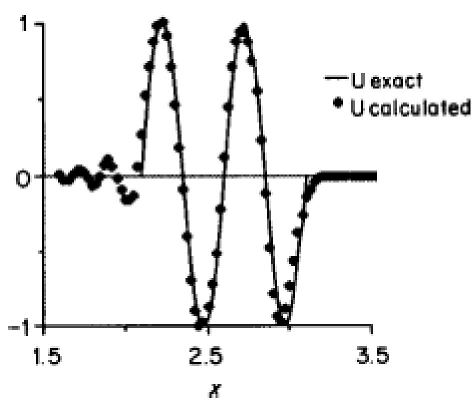
$$g = q\Delta t \pm \sqrt{1 + (q\Delta t)^2} \quad \rightarrow$$

$$\begin{aligned} g &= -I\frac{c\Delta t}{\Delta x}\sin\eta \pm \sqrt{1 - (\frac{c\Delta t}{\Delta x}\sin\eta)^2} \\ &= -I\nu\sin\eta \pm \sqrt{1 - \nu^2\sin^2\eta} \\ &= Ie^{\pm I(\frac{\pi}{2}+\alpha)} \text{ with } \sin\alpha = \nu\sin\eta \\ |g|^2 &= 1 \end{aligned}$$

And we observe that the first order upwind-forward Euler method is dissipative whereas the leapfrog method is not. It thus seems that the leapfrog method is in all ways (truncation error, dissipative properties) superior to the first order upwind forward Euler method. Let us check this conclusion by looking at numerical examples. We first consider the advection of a wave packet of period 0.5 ($k = 4\pi$) on a mesh of size $\Delta x = 1/40$ (hence $\eta = k\Delta x = \pi/10$), using a time step such that the CFL number $\nu = 0.8$. The numerical results obtained by both methods shown below confirm the conclusions of the analysis: the dissipative properties of the first order upwind-forward Euler method result in a serious reduction of the wave amplitude, whereas in contrast, the leapfrog solution almost perfectly agrees with the exact solution. We can however observe some trailing oscillations in the leapfrog solution.



Forward time upwind scheme



Leapfrog scheme