UNIT – II

VECTOR CALCULUS

Scalar Quantity :

A scalar quantity is that which has magnitude and is not related to any direction.

Vector Quantity :

A vector quantity is that which has both magnitude and direction.

Scalar Point function :

If corresponding to each point P of a region R there corresponds a scalar denoted by $\varphi(P)$ or $\varphi(x, y, z)$ then φ is said to be a scalar point function for the region R.

Example : The temperature $\varphi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector point function :

If corresponding to each point P of a region R, there corresponds a vector denoted by F(P), then F is said to be a vector point function for the region R.

Example : The acceleration F(P) of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential Operator:

The vector differential operator ∇ is defined as,

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \sum \vec{i}\frac{\partial}{\partial x}$$

Gradient of a scalar point function:

Let $\varphi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector,

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

is called the gradient of φ and is written as grad φ.

i.e., $\boxed{\text{grad } \varphi = \nabla\varphi}$

Note:

1. $\nabla\varphi$ defines a vector field.
2. $\nabla\varphi \neq \varphi\nabla$. There will be no '.' or 'x' between φ and ∇.

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

(or) $$\text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g$$

(2) If f and g are two scalar point functions then,

$$\nabla(fg) = f\nabla g + g\nabla f$$

(or) $$\text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$$

(3) If f and g are two scalar point functions then,

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0$$

(or) $$\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}$$

(4) Gradient of a constant is zero.

i.e., $\nabla\varphi = 0$

Problems :

(1) Find grad φ where $\varphi = x^2+y^2+z^2$.

Soln :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

(2) Find grad φ if $\varphi = xyz$ at $(1, 1, 1)$

Soln :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz$$

$$= \vec{i}\,yz + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.}$$

(3) Find grad φ where $\varphi = 3x^2y - y^3z^2$ at $(1,1,1)$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2)$$

$$+ \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}\,(6xy) + \vec{j}\,(3x^2 - 3y^2z^2) + \vec{k}\,(-2y^3z)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = 6\vec{i} - 2\vec{k}}$$

(4) If $\varphi = \log(x^2+y^2+z^2)$ find $\nabla\varphi$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}[\log(x^2+y^2+z^2)] + \vec{j}\frac{\partial}{\partial y}[\log(x^2+y^2+z^2)]$$

$$+ \vec{k}\frac{\partial}{\partial z}[\log(x^2+y^2+z^2)]$$

$$= \vec{i}\frac{1}{x^2+y^2+z^2}(2x) + \vec{j}\frac{1}{x^2+y^2+z^2}(2y) +$$

$$\vec{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$