



DEPARTMENT OF MATHEMATICS

Unit Normal :

A unit normal to the given surface ϕ at the point is $\frac{\nabla\phi}{|\nabla\phi|}$

Directional Derivative :

The directional derivative of ϕ in the direction \vec{a} is given by,

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} \quad (\text{or}) \quad \nabla\phi \cdot \hat{n} \quad \text{where} \quad \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\phi|$.

Angle between two surfaces :

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

Note :

If the surfaces cut orthogonally then,

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$



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Problems :

- ① Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$

Soln: $\phi : x^2y + 2xz - 4$

$$\begin{aligned}\nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (x^2y + 2xz - 4) + \vec{j} \frac{\partial}{\partial y} (x^2y + 2xz - 4) \\ &\quad + \vec{k} \frac{\partial}{\partial z} (x^2y + 2xz - 4) \\ &= \vec{i} (2xy + 2z) + \vec{j} (x^2) + \vec{k} (2x)\end{aligned}$$

$$\begin{aligned}\nabla\phi_{(2, -2, 3)} &= \vec{i} (-8 + 6) + \vec{j} (4) + \vec{k} (4) \\ &= -2\vec{i} + 4\vec{j} + 4\vec{k}\end{aligned}$$

$$|\nabla\phi| = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

Unit normal to the given surface at $(2, -2, 3)$

$$\begin{aligned}&= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6} \\ &= \frac{1}{3} (-\vec{i} + 2\vec{j} + 2\vec{k})\end{aligned}$$

- ② Find the unit vector normal to $x^2 - y^2 + z = 2$ at $(1, -1, 2)$.

Soln: $\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$

- ③ Find the unit vector normal to $x^2 + xy + z^2 = 4$ at $(1, -1, 2)$

Soln: $\frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$



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- 4) Find the directional derivative of the function $x^2 + 2xy$ at $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$

Soln:

$$\phi = x^2 + 2xy$$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x}(x^2 + 2xy) + \vec{j} \frac{\partial}{\partial y}(x^2 + 2xy) + \vec{k} \frac{\partial}{\partial z}(x^2 + 2xy)$$

$$= \vec{i}(2x + 2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\phi_{(1, -1, 3)} = \vec{i}(2 - 2) + \vec{j}(2) = 2\vec{j}$$

$$\text{Given: } \vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$\nabla\phi \cdot \hat{n} = 2\vec{j} \cdot \left[\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} \right] = \frac{4}{3}$$

$$\nabla\phi \cdot \hat{n} = \frac{4}{3}$$

- 5) Find the directional derivative of $xy + yz + zx$ at $(1, 1, 1)$ in the direction $\vec{i} + \vec{j}$.

Soln:

$$2\sqrt{2}$$

- 6) Find the directional derivative of $3x^2 + 2y - 3z$ at $(1, 1, 1)$ in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.

Soln:

$$\frac{19}{3}$$



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⑦ What is the greatest rate of increase of

$\phi = xyz^2$ at $(1, 0, 3)$?

Soln:

Let $\phi = xyz^2$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (xyz^2) + \vec{j} \frac{\partial}{\partial y} (xyz^2) + \vec{k} \frac{\partial}{\partial z} (xyz^2)$$

$$= \vec{i} (yz^2) + \vec{j} (xz^2) + \vec{k} (2xyz)$$

$$\nabla\phi_{(1,0,3)} = \vec{i} (0) + \vec{j} (9) + \vec{k} (0)$$

$$= 9\vec{j}$$

Maximum (or) Greatest rate of increase = $|\nabla\phi|$

$$= \sqrt{9^2}$$

$$= 9$$

⑧ In what direction from the point $(1, -1, 2)$ is the directional derivative of $\phi = x^2 y^2 z^3$ a maximum?

What is the magnitude of this maximum?

Soln:

$\phi = x^2 y^2 z^3$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 y^2 z^3) + \vec{j} \frac{\partial}{\partial y} (x^2 y^2 z^3) +$$

$$\vec{k} \frac{\partial}{\partial z} (x^2 y^2 z^3)$$

$$= 2xy^2 z^3 \vec{i} + 2x^2 y z^3 \vec{j} + 2x^2 y^2 z^2 \vec{k}$$

$\nabla\phi_{(1,-1,2)} = 16\vec{i} - 16\vec{j} + 12\vec{k}$ is the directional derivative.

$$\text{Magnitude is } |\nabla\phi| = \sqrt{16^2 + 16^2 + 12^2} = \sqrt{656}$$



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10) Find the angle between the surfaces $x^2 + y^2 + z^2 = 5$ and $x^2 + y^2 + z^2 - 2x = 5$ at $(0, 1, 2)$

Soln:

$$\text{Let } \phi_1 : x^2 + y^2 + z^2 - 5 \quad ; \quad \phi_2 = x^2 + y^2 + z^2 - 2x - 5$$

$$\frac{\partial \phi_1}{\partial x} = 2x$$

$$\frac{\partial \phi_2}{\partial x} = 2x - 2$$

$$\frac{\partial \phi_1}{\partial y} = 2y$$

$$\frac{\partial \phi_2}{\partial y} = 2y$$

$$\frac{\partial \phi_1}{\partial z} = 2z$$

$$\frac{\partial \phi_2}{\partial z} = 2z$$

$$\nabla \phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \quad ; \quad \nabla \phi_2 = (2x-2)\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla \phi_1 \Big|_{(0,1,2)} = 2\vec{j} + 4\vec{k}$$

$$\nabla \phi_2 \Big|_{(0,1,2)} = -2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$|\nabla \phi_1| = \sqrt{4+16} = \sqrt{20}$$

$$|\nabla \phi_2| = \sqrt{4+4+16} = \sqrt{24}$$

Angle between the surfaces,

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$= \frac{(2\vec{j} + 4\vec{k}) \cdot (-2\vec{i} + 2\vec{j} + 4\vec{k})}{\sqrt{20} \sqrt{24}}$$

$$= \frac{4 + 16}{\sqrt{20} \sqrt{24}} = \frac{20}{\sqrt{20} \sqrt{24}}$$

$$\cos \theta = \sqrt{\frac{5}{6}}$$

$$\theta = \cos^{-1} \sqrt{\frac{5}{6}}$$