VECTOR CALCULUS

Scalar Quantity:

A scalar quantity is that which has magnitude and is not related to any direction.

Vector Quantity:

A vector quantity is that which has both magnitude and direction.

Scalar point function:

If corresponding to each point P of a region R there corresponds a scalar denoted by $\varphi(P)$ or $\varphi(x, y, z)$ then φ is said to be a scalar point function for the region R.

Example: The temperature $\varphi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector point function:

If corresponding to each point P of a region R, there corresponds a vector denoted by $F(P)$, then F is said to be a vector point function for the region R.

Example: The acceleration $F(P)$ of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential Operator:

The vector differential operator ∇ is defined as,

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \sum \vec{i}\frac{\partial}{\partial x}}$$

Gradient of a scalar point function:

Let $\varphi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector

$$\boxed{\begin{aligned}\nabla\varphi &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi \\ &= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\end{aligned}}$$

is called the gradient of φ and is written as grad φ.

i.e., $\boxed{\text{grad}\,\varphi = \nabla\varphi}$

Note:

1. $\nabla\varphi$ defines a vector field.
2. $\nabla\varphi \neq \varphi\nabla$. There will be no '.' or 'x' between φ and ∇.

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\boxed{\begin{aligned} &\nabla(f \pm g) = \nabla f \pm \nabla g \\ (\text{or})\ &\text{grad}\,(f \pm g) = \text{grad}\,f \pm \text{grad}\,g\end{aligned}}$$

(2) If f and g are two scalar point functions then,

$$\boxed{\begin{array}{l} \nabla(fg) = f\nabla g + g\nabla f \\ \text{(or)}\ \text{grad}(fg) = f(\text{grad}\, g) + g(\text{grad}\, f) \end{array}}$$

(3) If f and g are two scalar point functions then,

$$\boxed{\begin{array}{l} \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0 \\ \text{(or)}\ \text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad}\, f) - f(\text{grad}\, g)}{g^2} \end{array}}$$

(4) Gradient of a constant is zero.

i.e., $\boxed{\nabla\varphi = 0}$

Problems :

(1) Find grad φ where $\varphi = x^2+y^2+z^2$.

Soln :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

(2) Find grad φ if $\varphi = xyz$ at $(1,1,1)$

Soln :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}\,yz + \vec{j}(xz) + \vec{k}(xy)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}.}$$

(3) Find grad φ where $\varphi = 3x^2y - y^3z^2$ at $(1,1,1)$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$$

$$\boxed{\nabla\varphi_{(1,1,1)} = 6\vec{i} - 2\vec{k}}$$

(4) If $\varphi = \log(x^2+y^2+z^2)$ find $\nabla\varphi$.

Soln:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}[\log(x^2+y^2+z^2)] + \vec{j}\frac{\partial}{\partial y}[\log(x^2+y^2+z^2)] + \vec{k}\frac{\partial}{\partial z}[\log(x^2+y^2+z^2)]$$

$$= \vec{i}\frac{1}{x^2+y^2+z^2}(2x) + \vec{j}\frac{1}{x^2+y^2+z^2}(2y) + \vec{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$