STOKE'S THEOREM :

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds$$

Where $\hat{n}$ is the unit normal vector at any point of S.

Note :

1. If $\vec{F}$ is irrotational, $\nabla \times \vec{F} = 0$

$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds = 0$ & hence $\vec{F}$ is Conservative.

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

PROBLEMS :

(1) Verify stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the xoy plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.

Soln :

By Stoke's theorem,

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}\, ds$$

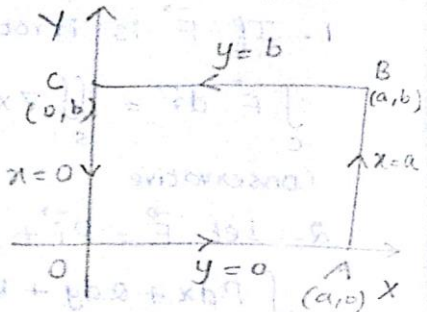
RHS :

Given : $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y) = 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$.



i.e., $\hat{n} = \vec{k}$

$\therefore$ Curl $\vec{F} \cdot \hat{n}\, ds$

$= 4y\vec{k} \cdot \vec{k}\, dx\, dy$

$= 4y\, dx\, dy$

$$\therefore \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \iint_S 4y\, dx\, dy$$

$$= \int_0^b \int_0^a 4y\, dx\, dy$$

$$= 4 \int_0^b [x]_0^a y\, dy$$

$$= 4a \left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \longrightarrow (1)$$

LHS :

$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2)\,dx + 2xy\,dy$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [(x^2 - y^2)\,dx + 2xy\,dy]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA ($y = 0$) :

$$\int_{OA} (x^2 - y^2)\,dx + 2xy\,dy = \int_0^a x^2\,dx \quad [\because y = 0 \Rightarrow dy = 0 \text{ Along } OA, x \to 0 \text{ to } a]$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

Along AB ($x = a$) :

$$\int_{AB} (x^2 - y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy \quad [\because x = 0 \Rightarrow dx = 0 \text{ Along } AB \; y \to 0 \text{ to } b]$$

$$= 2a\left[\frac{y^2}{2}\right]_0^b$$

$$= ab^2$$

Along BC ($y = b$) :

$$\int_{BC} (x^2 - y^2)\,dx + 2xy\,dy = \int_a^0 (x^2 - b^2)\,dx \quad [\because y = b \Rightarrow dy = 0 \text{ Along } BC \; x \to a \text{ to } 0]$$

$$= \left[\frac{x^3}{3} - b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} + ab^2$$

Along CO ($x=0$):

$$\int_{CO}(x^2-y^2)\,dx+2xy\,dy=\int_{CO}0+0 \qquad (\because x=0,\ dx=0)$$

Hence $$\int_C \vec{F}\cdot d\vec{r}=\int_{OA}+\int_{AB}+\int_{BC}+\int_{CO}$$

$$=\frac{a^3}{3}+ab^2-\frac{a^3}{3}+ab^2+0$$

$$=2ab^2 \rightarrow (2)$$

From (1) & (2),

$$\int_C \vec{F}\cdot d\vec{r}=\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Hence Stoke's theorem is verified.

(2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F}=y^2\vec{i}+x^2\vec{j}-(x+z)\vec{k}$ and C is the boundary of the triangle with vertices at (0,0,0), (1,0,0), (1,1,0).

Soln:

By Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r}=\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Since z coordinate is Zero in all the three vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F}=y^2\vec{i}+x^2\vec{j}-(x+z)\vec{k}$$