Unit Normal :

A unit normal to the given surface φ at the point is $\dfrac{\nabla\varphi}{|\nabla\varphi|}$

Directional Derivative :

The directional derivative of φ in the direction $\vec{a}$ is given by,

$$\nabla\varphi \cdot \frac{\vec{a}}{|\vec{a}|} \quad (\text{or}) \quad \nabla\varphi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\varphi|$.

Angle between two surfaces :

$$\boxed{\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}}$$

Note :

If the surfaces cut orthogonally then,

$$\boxed{\nabla\varphi_1 \cdot \nabla\varphi_2 = 0}$$

Problems :

(1) Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2,-2,3)$

Soln:

$$\varphi : x^2y + 2xz - 4$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y + 2xz - 4) + \vec{j}\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \vec{k}\frac{\partial}{\partial z}(x^2y + 2xz - 4)$$

$$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$$

$$\nabla\varphi_{(2,-2,3)} = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given surface at $(2,-2,3)$

$$= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

(2) Find the unit vector normal to $x^2 - y^2 + z = 2$ at $(1,-1,2)$.

Soln:

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

(3) Find the unit vector normal to $x^2 + xy + z^2 = 4$ at $(1,-1,2)$

Soln:

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$

(4) Find the directional derivative of the function x^2+2xy at $(1,-1,3)$ in the direction $\vec{i}+2\vec{j}+2\vec{k}$

Soln:

$$\varphi = x^2+2xy$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+2xy) + \vec{j}\frac{\partial}{\partial y}(x^2+2xy) + \vec{k}\frac{\partial}{\partial z}(x^2+2xy)$$

$$= \vec{i}(2x+2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\varphi_{(1,-1,3)} = \vec{i}(2-2) + \vec{j}(2) = 2\vec{j}$$

Given: $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$\nabla\varphi\cdot\hat{n} = 2\vec{j}\cdot\left[\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right] = \frac{4}{3}$$

$$\boxed{\nabla\varphi\cdot\hat{n} = \frac{4}{3}}$$

(5) Find the directional derivative of $xy+yz+zx$ at $(1,1,1)$ in the direction $\vec{i}+\vec{j}$.

Soln:

$$2\sqrt{2}$$

(6) Find the directional derivative of $3x^2+2y-3z$ at $(1,1,1)$ in the direction $2\vec{i}+2\vec{j}-\vec{k}$.

Soln:

$$\frac{19}{3}$$

(7) What is the greatest rate of increase of $\varphi = xyz^2$ at $(1,0,3)$?

Soln: Let $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,\frac{\partial}{\partial x}(xyz^2) + \vec{j}\,\frac{\partial}{\partial y}(xyz^2) + \vec{k}\,\frac{\partial}{\partial z}(xyz^2)$$

$$= \vec{i}\,(yz^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\nabla\varphi_{(1,0,3)} = \vec{i}\,(0) + \vec{j}\,(9) + \vec{k}\,(0)$$

$$= 9\vec{j}$$

Maximum (or) Greatest rate of increase $= |\nabla\varphi|$

$$= \sqrt{9^2}$$

$$= 9$$

(8) In what direction from the point $(1,-1,2)$ is the directional derivative of $\varphi = x^2y^2z^3$ a maximum? What is the magnitude of this maximum?

Soln: $\varphi = x^2y^2z^2$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,\frac{\partial}{\partial x}(x^2y^2z^2) + \vec{j}\,\frac{\partial}{\partial y}(x^2y^2z^2) + \vec{k}\,\frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= 2xy^2z^2\,\vec{i} + 2x^2yz^2\,\vec{j} + 2x^2y^2z\,\vec{k}$$

$\nabla\varphi_{(1,-1,2)} = 16\vec{i} - 16\vec{j} + 12\vec{k}$ is the directional derivative.

Magnitude is $|\nabla\varphi| = \sqrt{16^2+16^2+12^2} = \sqrt{656}$

(10) Find the angle between the surfaces $x^2+y^2+z^2=5$ and $x^2+y^2+z^2-2x=5$ at $(0,1,2)$

Soln:

Let $\varphi_1 : x^2+y^2+z^2-5$; $\varphi_2 = x^2+y^2+z^2-2x-5$

$\frac{\partial \varphi_1}{\partial x} = 2x$ $\qquad \frac{\partial \varphi_2}{\partial x} = 2x-2$

$\frac{\partial \varphi_1}{\partial y} = 2y$ $\qquad \frac{\partial \varphi_2}{\partial y} = 2y$

$\frac{\partial \varphi_1}{\partial z} = 2z$ $\qquad \frac{\partial \varphi_2}{\partial z} = 2z$

$\nabla\varphi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$; $\nabla\varphi_2 = (2x-2)\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\nabla\varphi_{1(0,1,2)} = 2\vec{j} + 4\vec{k}$ $\qquad \nabla\varphi_{2(0,1,2)} = -2\vec{i} + 2\vec{j} + 4\vec{k}$

$|\nabla\varphi_1| = \sqrt{4+16} = \sqrt{20}$ $\qquad |\nabla\varphi_2| = \sqrt{4+4+16} = \sqrt{24}$

Angle between the surfaces,

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$$= \frac{(2\vec{j}+4\vec{k})\cdot(-2\vec{i}+2\vec{j}+4\vec{k})}{\sqrt{20}\,\sqrt{24}}$$

$$= \frac{4+16}{\sqrt{20}\,\sqrt{24}} = \frac{20}{\sqrt{20}\,\sqrt{24}}$$

$$\cos\theta = \sqrt{\frac{5}{6}}$$

$$\boxed{\theta = \cos^{-1}\sqrt{\frac{5}{6}}}$$