



## DEPARTMENT OF MATHEMATICS

### LINE INTEGRALS :

Suppose  $c$  is an arc and  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  is the position vector of any point  $P(x, y, z)$  on it and  $\vec{f}$  is a vector point function at  $P$ . Then  $\int_c \vec{f} \cdot d\vec{r}$  is called a line integral of  $\vec{f}$  over  $c$ .

Line integral  $\int_A^B \vec{F} \cdot d\vec{r}$  is also known as the total work done by the force  $\vec{F}$  during a displacement from  $A$  to  $B$ .

- ① Evaluate  $\int_c \vec{F} \cdot d\vec{r}$  where  $\vec{F} = x^2 y^2 \vec{i} + y \vec{j}$  and the curve  $c$  is  $y^2 = 4x$  in the  $xy$ -plane from  $(0, 0)$  to  $(4, 4)$ .

Soln:

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\text{Given: } \vec{F} = x^2 y^2 \vec{i} + y \vec{j}$$

$$\vec{F} \cdot d\vec{r} = (x^2 y^2 \vec{i} + y \vec{j}) \cdot (dx\vec{i} + dy\vec{j})$$

$$= x^2 y^2 dx + y dy$$

$$\text{Given: } y^2 = 4x$$

$$2y dy = 4 dx$$

$$y dy = 2 dx$$

$$\therefore \vec{F} \cdot d\vec{r} = x^2 y^2 dx + 2 dx = x^2 (4x) dx + 2 dx$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2) dx$$



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$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2) dx$$
$$= \left[ \frac{4x^4}{4} + 2x \right]_0^4$$
$$= 4^4 + 8 = 256 + 8$$
$$= 264$$

② If  $\vec{F} = x^2 \vec{i} + xy \vec{j}$  evaluate  $\int_C \vec{F} \cdot d\vec{r}$  along the straight line  $y = x$  from  $(0,0)$  to  $(1,1)$ .  
Soln:  $2/3$ .

③ If  $\vec{F} = 5xy \vec{i} + 2y \vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $C$  is the part of the curve  $y = x^3$  between  $x=1$  and  $x=2$ .  
Soln:  $\vec{F} = 5xy \vec{i} + 2y \vec{j}$   
 $d\vec{r} = dx \vec{i} + dy \vec{j}$   
 $\vec{F} \cdot d\vec{r} = 5xy dx + 2y dy$   
Given:  $y = x^3 \Rightarrow dy = 3x^2 dx$   
 $\therefore \int_C \vec{F} \cdot d\vec{r} = \int_1^2 5x(x^3) dx + 2(x^3) 3x^2 dx$   
 $= \int_1^2 (5x^4 + 6x^5) dx$   
 $= \left[ \frac{5x^5}{5} + \frac{6x^6}{6} \right]_1^2$   
 $= [32 + 64 - (1 + 1)]$   
 $= 94$

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④ Find  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$

along the line joining the points  $(0,0,0)$  to  $(2,1,1)$ .

Soln:

$$\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (2y+3)dx + xzdy + (yz-x)dz$$

The equation of line joining the points  $(0,0,0)$  &  $(2,1,1)$  is

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$$x = 2t, \quad y = t, \quad z = t$$

$$\begin{aligned} \vec{F} \cdot d\vec{r} &= 2(2t+3)dt + 2t^2dt + (t^2-2t)dt \\ &= (3t^2 + 2t + 6)dt \end{aligned}$$

$$\text{At } x=0, y=0, z=0 \Rightarrow t=0$$

$$\text{At } x=2, y=1, z=1 \Rightarrow t=1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 2t + 6) dt$$

$$= [t^3 + t^2 + 6t]_0^1$$

$$= 8$$

⑤ If  $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$  evaluate  $\int_C \vec{F} \cdot d\vec{r}$  from  $(0,0,0)$  to  $(1,1,1)$  along the curve

$$x=t, \quad y=t^2, \quad z=t^3$$

Soln:



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$$\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx - 14yz dy + 20xz^2 dz$$

Given:  $x = t, y = t^2, z = t^3$

$$\Rightarrow dx = dt, dy = 2t dt, dz = 3t^2 dt$$

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= (3t^2 + 6t^2)dt - 14(t^3 \cdot t^3)2t dt + 20(t \cdot t^6)3t^2 dt \\ &= (9t^2 - 28t^6 + 60t^9) dt\end{aligned}$$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 (9t^2 - 28t^6 + 60t^9) dt \\ &= 5\end{aligned}$$

### SURFACE INTEGRAL:

Let  $S$  be a surface whose projection  $R_{xy}$  on the  $xy$  plane is such that the points on  $S$  have a 1-1 correspondence with the points on  $R_{xy}$ . Let  $ds$  be a vector element of the area. Then

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{xy}} \vec{F} \cdot \hat{n} \frac{dx dy}{|\hat{n} \cdot \vec{k}|}$$

$$\text{For } yz \text{ plane, } \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{yz}} \vec{F} \cdot \hat{n} \frac{dy dz}{|\hat{n} \cdot \vec{i}|}$$

$$\text{For } xz \text{ plane, } \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{xz}} \vec{F} \cdot \hat{n} \frac{dx dz}{|\hat{n} \cdot \vec{j}|}$$

The surface integral  $\iint_S \vec{F} \cdot d\vec{s}$  represents the total flux of  $\vec{F}$  through the whole surface.



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① Evaluate  $\iint_S \vec{F} \cdot \hat{n} \, ds$  where  $\vec{F} = z\vec{i} + x\vec{j} + 3y^2z\vec{k}$  and  $S$  is the surface of the cylinder  $x^2 + y^2 = 16$  included in the first octant between  $z=0$  &  $z=5$ .

Soln:

$$\vec{F} = z\vec{i} + x\vec{j} + 3y^2z\vec{k}$$

$$\phi = x^2 + y^2 - 16$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$\begin{aligned} \text{Now } \hat{n} &= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}} \\ &= \frac{x\vec{i} + y\vec{j}}{4} \quad (\because x^2 + y^2 = 16) \end{aligned}$$

Let us consider  $yz$ -plane

$z$  varies from 0 to 5

$y$  varies from 0 to 4

$$x^2 + y^2 = 16$$

$$\text{Put } x=0$$

$$y^2 = 16 \Rightarrow y = 4$$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{R_{xyz}} \vec{F} \cdot \hat{n} \frac{dy \, dz}{|\hat{n} \cdot \vec{i}|}$$

$$= \int_0^5 \int_0^4 (z\vec{i} + x\vec{j} + 3y^2z\vec{k}) \cdot \frac{(x\vec{i} + y\vec{j})}{4} \, dy \, dz$$

$$\frac{dy \, dz}{\left| \frac{x\vec{i} + y\vec{j}}{4} \cdot \vec{i} \right|}$$

$$= \int_0^5 \int_0^4 \left( \frac{zx}{4} + \frac{xy}{4} \right) \frac{dy \, dz}{\frac{x}{4}}$$

$$= \int_0^5 \int_0^4 \frac{x}{4} (z + y) \frac{dy \, dz}{x/4}$$