



Divergence of a vector point function.

Let \vec{F} be any given continuously differentiable vector point function then the divergence of \vec{F} is defined as,

$$\begin{aligned}\operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = (\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}) \cdot \vec{F} \\ &= \vec{i} \frac{\partial F_x}{\partial x} + \vec{j} \frac{\partial F_y}{\partial y} + \vec{k} \frac{\partial F_z}{\partial z}\end{aligned}$$

Note: 1. $\nabla \cdot \vec{F}$ is a scalar point function

2. If $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ be a continuously differentiable vector point function then,

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl of a vector point function:

Let $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ be any given continuously differentiable vector point function, the curl (or) rotation of \vec{F} is defined as,

$$\begin{aligned}\operatorname{Curl} \vec{F} &= \nabla \times \vec{F} = (\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}) \times \vec{F} \\ &= (\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}) \times (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}\end{aligned}$$

Note:- $\nabla \times \vec{F}$ is a vector point function.

Solenoidal vector:-

A vector \vec{F} is said to be solenoidal vector if $\operatorname{div} \vec{F} = 0$

Irrrotational vector:-

A vector \vec{F} is said to be irrotational if $\nabla \times \vec{F} = 0$

$$\text{i.e. } \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Conservative vector field:

If a vector point function \vec{F} is expressible as the gradient of a scalar point function ϕ , then \vec{F} is conservative. i.e. \vec{F} is conservative if $\vec{F} = \nabla \phi$. Here ϕ is called scalar potential. \vec{F} is conservative force if $\text{curl } \vec{F} = 0$.

Problems:

1. Prove that $\text{curl } (\nabla \phi) = 0$ (or) $\nabla \times \nabla \phi = 0$

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\text{curl } (\nabla \phi) = \nabla \times \nabla \phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right] - \vec{j} \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right] + \vec{k} \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$= 0.$$

d. Prove that $\text{div}(\text{curl } \vec{F}) = 0$ (or) $\nabla \cdot (\nabla \times \vec{F}) = 0$.

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \vec{j} \left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \vec{k} \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right]$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x} \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \frac{\partial}{\partial y} \left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \frac{\partial}{\partial z} \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

3. S.T. $\text{curl grad } f = 0$ (or) $\nabla \times \nabla f = 0$.

$$\text{Curl grad } f = \nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] - \vec{j} \left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right] + \vec{k} \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right]$$

$$= 0$$

4. If $\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$. What is the directional derivative

of v at the point $(1, 2, 3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$

$$\nabla v = y\vec{i} + z\vec{j} + x\vec{k} \quad \nabla v(1, 2, 3) = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k} \Rightarrow |\vec{a}| = \sqrt{25 + 16 + 9} = \sqrt{50}$$

$$\text{Directional derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|} = (2\vec{i} + 3\vec{j} + \vec{k}) \cdot \frac{(3\vec{i} + 4\vec{j} + 5\vec{k})}{\sqrt{50}}$$

$$= \frac{6 + 12 + 5}{\sqrt{50}}$$

$$= \frac{23}{\sqrt{50}}$$

5. Prove that $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}$

$$\begin{aligned}\text{div}(\vec{u} \times \vec{v}) &= \sum_i \frac{\partial}{\partial x_i} (\vec{u} \times \vec{v})_i \\ &= \sum_i \left[\vec{u} \times \frac{\partial \vec{v}}{\partial x_i} + \frac{\partial \vec{u}}{\partial x_i} \times \vec{v} \right] \\ &= \sum_i \left(\frac{\partial \vec{u}}{\partial x_i} \times \vec{v} \right) + \sum_i \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x_i} \right) \\ &= \left(\sum_i \frac{\partial \vec{u}}{\partial x_i} \right) \cdot \vec{v} - \left(\sum_i \frac{\partial \vec{v}}{\partial x_i} \right) \cdot \vec{u} \\ &= \text{curl} \vec{u} \cdot \vec{v} - \text{curl} \vec{v} \cdot \vec{u} \\ &= \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}.\end{aligned}$$