

Solenoidal vector :-

A vector \vec{F} is said to be solenoidal vector if $\text{div} \vec{F} = 0$

Irrrotational vector :-

A vector \vec{F} is said to be irrotational if $\nabla \times \vec{F} = 0$

$$\text{i.e. } \text{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Conservative vector field:

If a vector point function \vec{F} is expressible as the gradient of a scalar point function ϕ , then \vec{F} is conservative i.e. \vec{F} is conservative if $\vec{F} = \nabla \phi$. Here ϕ is called scalar potential. \vec{F} is conservative force if $\text{curl} \vec{F} = 0$.



5. Prove that $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}$

$$\begin{aligned} \text{div}(\vec{u} \times \vec{v}) &= \sum \vec{i} \frac{\partial}{\partial x} (\vec{u} \times \vec{v}) \\ &= \sum \vec{i} \left[\vec{u} \times \frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x} \times \vec{v} \right] \\ &= \sum \vec{i} \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} \right) + \sum \vec{i} \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x} \right) \\ &= \left(\sum \vec{i} \times \frac{\partial \vec{u}}{\partial x} \right) \cdot \vec{v} + \left(\sum \vec{i} \times \frac{\partial \vec{v}}{\partial x} \right) \cdot \vec{u} \\ &= \text{curl} \vec{u} \cdot \vec{v} - \text{curl} \vec{v} \cdot \vec{u} \\ &= \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v} \end{aligned}$$

6. Prove that $r^n \vec{r}$ is solenoidal if $n = -3$ & $r^n \vec{r}$ is irrotational for all values of n .

$$\begin{aligned} r^n \vec{r} &= r^n (x\vec{i} + y\vec{j} + z\vec{k}) \\ &= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k} \end{aligned}$$

$$\begin{aligned} \text{div}(r^n \vec{r}) &= \nabla \cdot (r^n \vec{r}) \\ &= \frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z) \end{aligned}$$

$$\text{Now } r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} ; 2r \frac{\partial r}{\partial y} = 2y ; 2r \frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now, } \frac{\partial}{\partial x} (r^n x) = x(nr^{n-1}) \frac{\partial r}{\partial x} + r^n$$

$$= xnr^{n-1} \frac{x}{r} + r^n = x^2 nr^{n-2} + r^n$$

$$\text{Similarly } \frac{\partial}{\partial y} (r^n y) = y^2 nr^{n-2} + r^n \text{ \& } \frac{\partial}{\partial z} (r^n z) = z^2 nr^{n-2} + r^n$$

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(AN AUTONOMOUS INSTITUTION)
COIMBATORE - 35
DEPARTMENT OF MATHEMATICS

$$\begin{aligned} \operatorname{div} (r^n \vec{r}) &= x^2 n r^{n-2} + r^n + y^2 n r^{n-2} + r^n + z^2 n r^{n-2} + r^n \\ &= 3r^n + n r^{n-2} (x^2 + y^2 + z^2) \\ &= 3r^n + n r^{n-2} (r^2) \\ &= 3r^n + n r^n = (3+n)r^n \end{aligned}$$

The vector $r^n \vec{r}$ is solenoidal if, $\operatorname{div} (r^n \vec{r}) = 0$

$$\Rightarrow (n+3)r^n = 0$$

$$n+3 = 0 \Rightarrow n = -3$$

$\therefore r^n \vec{r}$ is solenoidal if $n = -3$

$$\operatorname{curl} (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \vec{i} \left[\frac{\partial}{\partial y} r^n z - \frac{\partial}{\partial z} r^n y \right]$$

$$= \sum \vec{i} \left[n r^{n-1} \frac{\partial r}{\partial y} z - n r^{n-1} \frac{\partial r}{\partial z} y \right]$$

$$= \sum \vec{i} \left[n r^{n-1} \frac{y}{r} z - n r^{n-1} \frac{z}{r} y \right]$$

$$= \sum \vec{i} n r^{n-2} [yz - zy]$$

$$= 0$$

$\therefore \operatorname{curl} (r^n \vec{r}) = 0$ for all values of n

14. Find the constant a, b, c if the vector

$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

$\therefore \vec{F}$ is irrotational $\Rightarrow \nabla \times \vec{F} = 0$.

$$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$

\Rightarrow Each component is equal to zero

$$\begin{array}{l|l|l} c+1=0 & 4-a=0 & b-2=0 \\ \Rightarrow c=-1 & \Rightarrow a=4 & \Rightarrow b=2 \end{array}$$

8) Find 'a' so that the vector $\vec{A} = (ax^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

is irrotational

$$\therefore \vec{A} \text{ is irrotational} \Rightarrow \nabla \times \vec{A} = 0$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2 - y^2 + x & -2xy - y & 0 \end{vmatrix} = 0$$

$$\vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y+2y] = 0$$

\therefore There is no 'a' value in the above equation

'a' is arbitrary.

9. Prove $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential ϕ $\Rightarrow \vec{F} = \nabla \phi$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2y \cos x - 2y \cos x] = 0$$

$\Rightarrow \vec{F}$ is irrotational

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COIMBATORE - 35
DEPARTMENT OF MATHEMATICS

To find ϕ

$$\nabla\phi = (y^2 \cos x + z^2)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$$

WRT, $\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y^2 \cos x$$

$$\phi = y^2 \sin x + xz^3 + f(y,z) \quad \left| \begin{array}{l} \frac{\partial\phi}{\partial y} = 2yz^3 \\ \frac{\partial\phi}{\partial z} = 3xz^2 \end{array} \right. \quad \left| \begin{array}{l} \frac{\partial\phi}{\partial y} = 2yz^3 \\ \frac{\partial\phi}{\partial z} = 3xz^2 \end{array} \right. \quad \left| \begin{array}{l} \frac{\partial\phi}{\partial y} = 2yz^3 \\ \frac{\partial\phi}{\partial z} = 3xz^2 \end{array} \right.$$

$$\phi = y^2 \sin x + xz^3 - 4y + C$$

Q. 11) S.T $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

is irrotational. Find ϕ $\Rightarrow \vec{F} = \nabla\phi$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} \quad \nabla \times \vec{F} = 0$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = 0$$

$\Rightarrow \nabla \times \vec{F}$ is irrotational

To find ϕ :

$$\vec{F} = \nabla\phi \Rightarrow (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy + z^3$$

$$\Rightarrow \phi = 6y \frac{x^2}{2} + z^3 x + f(y,z)$$

$$\Rightarrow \phi = 3x^2 y + xz^3 + f(y,z)$$

$$\left| \frac{\partial\phi}{\partial y} = 3x^2 - z \right.$$

$$\left| \phi = 3x^2 y - zy + f(y,z) \right.$$

$$\left| \frac{\partial\phi}{\partial z} = 3xz^2 - y \right.$$

$$\left| \phi = 3x \frac{z^3}{3} - yz + f(x,y) \right.$$

$$\left| \phi = xz^3 - yz + f(x,y) \right.$$

$$\phi = 3x^2 y + xz^3 - yz + C$$

11. Show that the vector $\vec{F} = 3y^4 z^2 \vec{i} + 4x^3 z \vec{j} - 3xy^2 z \vec{k}$

is solenoidal.

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DEPARTMENT OF MATHEMATICS

Given:- $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$

To prove: $\text{div}\vec{F} = 0$, $\nabla \cdot \vec{F} = 0$

$$\text{ie) } \text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0.$$

$\therefore \vec{F}$ is solenoidal.

12) Prove that $\text{div } \hat{r} = 2/r$

$$\begin{aligned} \text{div } \hat{r} &= \nabla \cdot \left(\frac{\vec{r}}{r} \right) = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r} \right) \\ &= \frac{1}{r} - \frac{1}{r^2} \cdot x \frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2} y \frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2} z \frac{\partial r}{\partial z} \\ &= \frac{3}{r} - \frac{1}{r^2} \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right] \end{aligned}$$

Now $r^2 = x^2 + y^2 + z^2$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}; \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \text{Now } \text{div } \hat{r} &= \frac{3}{r} - \frac{1}{r^2} \left[x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r} \right] \\ &= \frac{3}{r} - \frac{1}{r^2} \left[\frac{x^2 + y^2 + z^2}{r} \right] = \frac{3}{r} - \frac{1}{r^2} \cdot \frac{r^2}{r} = \frac{3}{r} - \frac{1}{r} \end{aligned}$$

$$\text{div } \hat{r} = \frac{2}{r}$$

13. P.T (curl curl \vec{F}) = $\nabla(\text{div } \vec{F}) - \nabla^2 \vec{F}$

Given: $\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F}) \nabla - (\nabla \cdot \nabla) \vec{F}$

$$[\because \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}]$$

$$= \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$= \nabla(\text{div } \vec{F}) - \nabla^2 \vec{F}$$