



Gauss Divergence Theorem:

If \vec{F} is a vector point function, finite and differentiable in a region R bounded by a closed surface S , then the surface integral of the normal component of \vec{F} taken over S is equal to the integral of divergence of \vec{F} taken over V .

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv.$$

where \hat{n} is the unit vector in the +ve (outward direction) normal to S .

1. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv.$$

$$\text{RHS: } \vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla \cdot \vec{F} = (2x) + (2y) + (2z)$$

$$= 2(x + y + z)$$

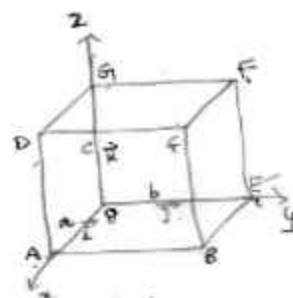
$$\iiint_V \nabla \cdot \vec{F} dv = \int_0^a \int_0^b \int_0^c 2(x + y + z) dz dy dx$$

$$= 2 \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right]_0^c dy dx$$

$$= 2 \int_0^a \int_0^b \left[cx + cy + \frac{c^2}{2} \right] dy dx$$

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$$\begin{aligned}
 &= 2 \int_0^a \left[cxy + \frac{cy^2}{2} + \frac{c^2}{2}y \right]_0^b dx \\
 &= 2 \int_0^a \left[bcx + \frac{b^2c}{2} + \frac{bc^2}{2} \right] dx \\
 &= 2 \left[\frac{bcx^2}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2} \right]_0^a \\
 &= 2 \left[\frac{a^2bc + ab^2c + abc^2}{2} \right] \\
 &= abc [a+b+c] \rightarrow \text{①}
 \end{aligned}$$



LHS: $\iint_S \vec{F} \cdot \hat{n} ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	\hat{n}	$\vec{F} \cdot \hat{n}$	ds	Face eqn	$\vec{F} \cdot \hat{n}$ on S.
S_1 -ABCD	\hat{i}	$x^2 - yz$	$dydz$	$x=a$	$a^2 - yz$
S_2 -DEFG	$-\hat{i}$	$y^2 - xz$	$dydz$	$x=0$	y^2
S_3 -BCEF	\hat{j}	$y^2 - zx$	$dx dz$	$y=b$	$b^2 - zx$
S_4 -ADGH	$-\hat{j}$	$zx - y^2$	$dx dz$	$y=0$	zx
S_5 -DCGF	\hat{k}	$z^2 - xy$	$dx dy$	$z=c$	$c^2 - xy$
S_6 -OABE	$-\hat{k}$	$xy - z^2$	$dx dy$	$z=0$	xy

$$\begin{aligned}
 \iint_{S_1} \vec{F} \cdot \hat{n} ds &= \int_0^c \int_0^b (a^2 - yz) dy dz = \int_0^c \left(a^2y - \frac{y^2}{2}z \right) \Big|_0^b dz \\
 &= \int_0^c \left(a^2b - \frac{b^2z}{2} \right) dz = \left(a^2bz - \frac{b^2z^2}{4} \right) \Big|_0^c \\
 &= a^2bc - \frac{b^2c^2}{4}
 \end{aligned}$$

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$$\iint_{S_2} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^b yz \, dy \, dz = \int_0^c \left[\frac{y^2}{2} z \right]_0^b dz$$

$$= \int_0^c \left(\frac{b^2}{2} z \right) dz = \left(\frac{b^2 z^2}{4} \right)_0^c = \frac{b^2 c^2}{4}$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^a (b^2 - zx) \, dx \, dz = \int_0^c \left(b^2 x - \frac{zx^2}{2} \right)_0^a dz$$

$$= \int_0^c \left(b^2 a - \frac{za^2}{2} \right) dz = \left(b^2 az - \frac{za^2}{4} \right)_0^c$$

$$= ab^2 c - \frac{a^2 c^2}{4}$$

$$\iint_{S_4} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^a zx \, dx \, dz = \int_0^c \left(\frac{zx^2}{2} \right)_0^a dz = \int_0^c \frac{a^2 z}{2} dz$$

$$= \left[\frac{a^2 z^2}{4} \right]_0^c = \frac{a^2 c^2}{4}$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a (c^2 - xy) \, dx \, dy = \int_0^b \left(c^2 x - \frac{xy^2}{2} \right)_0^a dy$$

$$= \int_0^b \left(c^2 a - \frac{a^2 y}{2} \right) dy = \left(ac^2 y - \frac{a^2 y^2}{4} \right)_0^b$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

$$\iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a xy \, dx \, dy = \int_0^b \left(\frac{x^2 y}{2} \right)_0^a dy = \int_0^b \frac{a^2 y}{2} dy$$

$$= \left(\frac{a^2 y^2}{4} \right)_0^b = \frac{a^2 b^2}{4}$$

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$$\begin{aligned} \therefore \iint_S \vec{F} \cdot \hat{n} ds &= a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{a^2c^2}{4} \\ &\quad + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4} \\ &= a^2bc + ab^2c + abc^2 \\ &= abc(a+b+c) \rightarrow \textcircled{2} \end{aligned}$$

From ① & ②

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

2. Verify divergence theorem for $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} + xz^2\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

$$\text{RHS} \rightarrow \vec{F} = (2x-z)\vec{i} + x^2y\vec{j} + xz^2\vec{k}$$

$$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$$

$$\begin{aligned} \iiint_V \nabla \cdot \vec{F} dv &= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz) dx dy dz \\ &= \int_0^1 \int_0^1 \left(2x + \frac{x^3}{3} - \frac{2x^2z}{2} \right) dy dz \\ &= \int_0^1 \int_0^1 (2 + y^3 - z) dy dz = \int_0^1 \left[2y + \frac{y^4}{3} - \frac{zy}{1} \right]_0^1 dz \\ &= \int_0^1 \left(2 + \frac{1}{3} - z \right) dz = \left(\frac{7}{3}z - \frac{z^2}{2} \right) \Big|_0^1 \\ &= \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6} \end{aligned}$$

$$\iiint_V \nabla \cdot \vec{F} dv = \frac{11}{6} \rightarrow \textcircled{1}$$

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$$\text{LHS} \Rightarrow \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	\hat{n}	$\vec{F} \cdot \hat{n}$	ds	Face eqn	$\vec{F} \cdot \hat{n}$ on S
S_1 -ABCD	\vec{i}	$2x-z$	$dydz$	$x=1$	$2-z$
S_2 -DEFG	$-\vec{i}$	$z-2x$	$dydz$	$x=0$	z
S_3 -BCFE	\vec{j}	x^2y	$dx dz$	$y=1$	x^2
S_4 -OADG	$-\vec{j}$	$-x^2y$	$dx dz$	$y=0$	0
S_5 -DCFG	\vec{k}	$-x^2z$	$dx dy$	$z=1$	$-x$
S_6 -OABE	$-\vec{k}$	x^2z	$dx dy$	$z=0$	0

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot \hat{n} \, ds &= \int_0^1 \int_0^1 (2-z) \, dy \, dz = \int_0^1 (2y - yz) \Big|_0^1 \, dz = \int_0^1 (2-z) \, dz \\ &= \left[2z - \frac{z^2}{2} \right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2} \end{aligned}$$

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 z \, dy \, dz = \int_0^1 (zy) \Big|_0^1 \, dz = \int_0^1 z \, dz = \left[\frac{z^2}{2} \right]_0^1 = \frac{1}{2}$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 x^2 \, dx \, dz = \int_0^1 \left(\frac{x^3}{3} \right) \Big|_0^1 \, dz = \int_0^1 \frac{1}{3} \, dz = \left[\frac{z}{3} \right]_0^1 = \frac{1}{3}$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 -x \, dx \, dy = \int_0^1 \left(-\frac{x^2}{2} \right) \Big|_0^1 \, dy = \int_0^1 \left(-\frac{1}{2} \right) \, dy = \left[-\frac{y}{2} \right]_0^1 = -\frac{1}{2}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \frac{3}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \frac{3}{2} + \frac{1}{3}$$

$$= \frac{9+2}{6} = \frac{11}{6} \rightarrow \textcircled{2} \quad \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV$$

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Q. Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm 1, y = \pm 1, z = \pm 1$

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

RHS $\Rightarrow \vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 2x + y$$

$$\iiint_V \nabla \cdot \vec{F} \, dv = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + y) \, dz \, dy \, dx$$

$$= \int_{-1}^1 \int_{-1}^1 [2xz + yz]_{-1}^1 \, dy \, dx = \int_{-1}^1 \int_{-1}^1 [(2x+y) - (-2x-y)] \, dy \, dx$$

$$= 2 \int_{-1}^1 \int_{-1}^1 (2x+y) \, dy \, dx = 2 \int_{-1}^1 \left[2xy + \frac{y^2}{2} \right]_{-1}^1 \, dx$$

$$= 2 \int_{-1}^1 [(2x + 1/2) - (-2x + 1/2)] \, dx = 2 \int_{-1}^1 4x \, dx$$

$$= 8 \left[\frac{x^2}{2} \right]_{-1}^1 = 4(1-1) = 0 \rightarrow \text{①}$$

LHS $\Rightarrow \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	\hat{n}	$\vec{F} \cdot \hat{n}$	ds	Face eqn	$\vec{F} \cdot \hat{n}$ on ds
S_1 - ABCD	\vec{i}	x^2	$dydz$	$x=1$	1
S_2 - OCFG	$-\vec{i}$	$-x^2$	$dydz$	$x=-1$	-1
S_3 - BCFE	\vec{j}	z	$dx dz$	$y=1$	z
S_4 - OADG	$-\vec{j}$	$-z$	$dx dz$	$y=-1$	-z
S_5 - DCFG	\vec{k}	yz	$dx dy$	$z=1$	y
S_6 - OABE	$-\vec{k}$	$-yz$	$dx dy$	$z=-1$	y

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$$\iint_{S_1} \vec{F} \cdot \hat{n} \, ds = \int_{-1}^1 \int_{-1}^1 dy \, dz = \int_{-1}^1 [y]_{-1}^1 dz = \int_{-1}^1 (1+1) dz = 2 \int_{-1}^1 dz = 4$$

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, ds = - \int_{-1}^1 \int_{-1}^1 dy \, dz = - \int_{-1}^1 [y]_{-1}^1 dz = -2 \int_{-1}^1 dz = -2[z]_{-1}^1 = -4$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \int_{-1}^1 \int_{-1}^1 z \, dx \, dz = \int_{-1}^1 [zx]_{-1}^1 dz = 2 \int_{-1}^1 z \, dz = 2 \left[\frac{z^2}{2} \right]_{-1}^1 = 0$$

$$\iint_{S_4} \vec{F} \cdot \hat{n} \, ds = \int_{-1}^1 \int_{-1}^1 -z \, dx \, dz = \int_{-1}^1 [-zx]_{-1}^1 dz = -2 \int_{-1}^1 z \, dz = 0$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_{-1}^1 \int_{-1}^1 y \, dx \, dy = 2 \int_{-1}^1 y \, dy = 2 \left[\frac{y^2}{2} \right]_{-1}^1 = 0$$

$$\iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \int_{-1}^1 \int_{-1}^1 y \, dx \, dy = 0$$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = 4 - 4 + 0 + 0 + 0 + 0 = 0 \rightarrow \textcircled{2}$$

From $\textcircled{1}$ & $\textcircled{2}$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

✓ Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - yz\vec{j} + yz\vec{k}$

over the cube $x=0, x=1, y=0, y=1, z=0, z=1$

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

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$$\text{RHS} \Rightarrow \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\iiint_V \nabla \cdot \vec{F} \, dv = \iiint_V (4z - y) \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^1 [4xz - yx]_0^1 \, dy \, dz = \int_0^1 \int_0^1 (4z - y) \, dy \, dz$$

$$= \int_0^1 [4zy - \frac{y^2}{2}]_0^1 \, dz = \int_0^1 [4z - \frac{1}{2}] \, dz$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z \right]_0^1 = \left[2z^2 - \frac{1}{2}z \right]_0^1$$

$$= [2 - \frac{1}{2}] = \frac{3}{2} \rightarrow \text{①}$$

LHS \Rightarrow

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	\hat{n}	$\vec{F} \cdot \hat{n}$	ds	Face eqn	$\vec{F} \cdot \hat{n}$ on S
$S_1 - ABCD$	\hat{i}	$4xz$	$dy \, dz$	$x=1$	$4z$
$S_2 - O EFG$	$-\hat{i}$	$-4xz$	$dy \, dz$	$x=0$	0
$S_3 - BCFE$	\hat{j}	$-y^2$	$dx \, dz$	$y=1$	-1
$S_4 - OMDG$	$-\hat{j}$	y^2	$dx \, dz$	$y=0$	0
$S_5 - DCFG$	\hat{k}	yz	$dx \, dy$	$z=1$	y
$S_6 - OABE$	$-\hat{k}$	$-yz$	$dx \, dy$	$z=0$	0

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$$\begin{aligned}\iint_{S_1} \vec{F} \cdot \hat{n} \, ds &= \int_0^1 \int_0^1 4z \, dy \, dz = \int_0^1 (4yz)'_0^1 \, dz = \int_0^1 4z \, dz \\ &= \left[\frac{4z^2}{2} \right]_0^1 = (2z^2)'_0^1 = 2\end{aligned}$$

$$\begin{aligned}\iint_{S_3} \vec{F} \cdot \hat{n} \, ds &= - \int_0^1 \int_0^1 dx \, dy = - \int_0^1 (x)'_0^1 \, dy = - \int_0^1 dy = - [y]'_0^1 \\ &= -1\end{aligned}$$

$$\begin{aligned}\iint_{S_5} \vec{F} \cdot \hat{n} \, ds &= \int_0^1 \int_0^1 y \, dz \, dy = \int_0^1 [zy]'_0^1 \, dy = \int_0^1 y \, dy = \left[\frac{y^2}{2} \right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} \, ds &= \iint_{S_1} \vec{F} \cdot \hat{n} \, ds + \iint_{S_2} \vec{F} \cdot \hat{n} \, ds + \iint_{S_5} \vec{F} \cdot \hat{n} \, ds \\ &= 2 - 1 + \frac{1}{2} \\ &= 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$