Continuous Two Dimensional Random Variables.

1. Given $f(x,y) = \begin{cases} cx(x-y), & 0 < x < 2, -x < y < x \\ 0, & otherwise \end{cases}$

Find i). c ii). $f(x)$ iii). $f(y/x)$

Soln.

i).
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dy \, dx = 1$$

$$\int_{0}^{2} \int_{-x}^{x} cx(x-y) \, dy \, dx = 1$$

$$c \int_{0}^{2} \int_{-x}^{x} (x^2 - xy) \, dy \, dx = 1$$

$$c \int_{0}^{2} \left[x^2 y - x \frac{y^2}{2} \right]_{y=-x}^{x} dx = 1$$

$$c \int_{0}^{2} \left[\left(x^3 - \frac{x^3}{2} \right) - \left(-x^3 - \frac{x^3}{2} \right) \right] dx = 1$$

$$c \int_{0}^{2} \left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2} \right] dx = 1$$

$$c \int_{0}^{2} 2x^3 \, dx = 1$$

$$2c \left(\frac{x^4}{4} \right)_{0}^{2} = 1$$

$$\frac{c}{2} (2^4 - 0) = 1$$

$$\frac{16c}{2} = 1$$

$$8c = 1 \Rightarrow c = 1/8$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8} x(x-y), & 0 < x < 2, \ -x < y < x \\ 0, & \text{otherwise.} \end{cases}$$

ii). Marginal density function of x (MDF OF x)

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-x}^{x} \frac{1}{8} x(x-y)\, dy$$

$$= \frac{1}{8} \int_{-x}^{x} (x^2 - xy)\, dy$$

$$= \frac{1}{8} \left[x^2 y - x \frac{y^2}{2} \right]_{y=-x}^{x}$$

$$= \frac{1}{8} \left[\left(x^3 - \frac{x^3}{2} \right) - \left(-x^3 - \frac{x^3}{2} \right) \right]$$

$$= \frac{1}{8} \left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2} \right]$$

$$= \frac{2x^3}{8}$$

$$f(x) = \frac{x^3}{4}, \quad 0 < x < 2$$

iii). $f(y/x)$

WKT $\quad f(y/x) = \dfrac{f(x,y)}{f(x)}$

$$= \frac{\frac{1}{8} x(x-y)}{x^3/4}$$

$$= \frac{4/8(x^2 - xy)}{x^3}$$

$$= \frac{1}{2} \frac{x(x-y)}{x^3}$$

$$F(Y/x) = \frac{x-y}{2x^2}$$

2]. The Joint Probability density function

$$F(x,y) = \begin{cases} xy^2 + \dfrac{x^2}{8}, & 0 \le x \le 2 \ \& \ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

i). $P(x>1/y<\frac{1}{2})$ ii). $P(y<\frac{1}{2}/x>1)$

iii). $P(x<y)$ iv). $P(x+y \le 1)$

Soln.

i). $P(x>1/y<\frac{1}{2})$

$$= \frac{P(x>1, \ y<\frac{1}{2})}{P(y<\frac{1}{2})} \qquad \rightarrow (1)$$

Now, $P[x>1, \ y<\frac{1}{2}] = \int\limits_{1}^{2}\int\limits_{0}^{1/2}(xy^2 + \frac{x^2}{8})\,dy\,dx$

$$= \int\limits_{1}^{2} \left[x\frac{y^3}{3} + \frac{x^2}{8}y \right]_{y=0}^{1/2} dx$$

$$= \int\limits_{1}^{2} \left\{ \left(\frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\left(\frac{1}{2}\right) \right) - 0 \right\} dx$$

$$= \int\limits_{1}^{2} \left(\frac{x}{24} + \frac{x^2}{16} \right) dx$$

$$= \left(\frac{1}{24} \frac{x^2}{2} + \frac{1}{16} \frac{x^3}{3} \right)_{1}^{2}$$

$$= \left(\frac{4}{48} + \frac{8}{48} \right) - \left(\frac{1}{48} + \frac{1}{48} \right)$$

$$= \frac{12}{48} - \frac{2}{48}$$

$$= \frac{10}{48}$$

$$= \frac{5}{24} \longrightarrow (2)$$

ii). $P\left(y < \frac{1}{2} \;/\; x > 1 \right)$

$$= \frac{P(x>1, \; y < \frac{1}{2})}{P(x>1)}$$

$$P\left(y < \frac{1}{2} \right) = \int_{0}^{2} \int_{0}^{1/2} F(x, y) \; dy \; dx$$

$$= \int_{0}^{2} \int_{0}^{1/2} \left(xy^2 + \frac{x^2}{8} \right) dy \; dx$$

$$= \int_{0}^{2} \left[\frac{xy^3}{3} + \frac{x^2}{8} y \right]_{y=0}^{1/2} dx$$

$$= \int_{0}^{2} \left[\frac{x}{3} \left(\frac{1}{8} \right) + \frac{x^2}{8} \left(\frac{1}{2} \right) \right] dx$$

$$= \int_{0}^{2} \left[\frac{x}{24} + \frac{x^2}{16} \right] dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48} \right]_{0}^{2}$$

$$= \frac{4}{48} + \frac{8}{48} = \frac{12}{48}$$

(1) $\Rightarrow P\left(x > 1 \mid y < \frac{1}{2}\right) = \frac{5}{24} \cdot \frac{4}{1}$

$$= \frac{5}{6}$$

ii). $P\left(y < \frac{1}{2} \mid x > 1\right)$

$$= \frac{P\left(x > 1, \ y < \frac{1}{2}\right)}{P(x > 1)} \longrightarrow (3)$$

Now

$$P(x > 1) = \int_{1}^{2}\int_{0}^{1} f(x, y) \, dy \, dx$$

$$= \int_{1}^{2}\int_{0}^{1} \left(xy^2 + \frac{x^2 y}{8}\right) dy \, dx$$

$$= \int_{1}^{2} \left[\frac{x y^3}{3} + \frac{x^2}{8} \frac{y^2}{2}\right]_{0}^{1} dx$$

$$= \int_{1}^{2} \left[\frac{x}{3} + \frac{x^2}{8}\right] dx$$

$$= \left(\frac{x^2}{6} + \frac{x^3}{24}\right)_{1}^{2}$$

$$= \left(\frac{4}{6} + \frac{8}{24}\right) - \left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= \frac{16 + 8 - 4 - 1}{24}$$

$$= \frac{19}{24}$$

(3) $\Rightarrow P\left(y < \frac{1}{2} \mid x > 1\right) = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$

ii) $P(x < y)$

$$= \int_0^1 \int_0^y f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96} \right]_0^1 \qquad \frac{24 \times 4}{96}$$

$$= \frac{1}{10} + \frac{1}{96}$$

$$= \frac{96 + 10}{960} = \frac{106}{960}$$

iii) $P(x + y \leq 1)$

$$= \int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right] dy$$

$$= \int_0^1 \left[\frac{(1+y^2-2y)\, y^2}{2} + \frac{(1-y)(1-y)^2}{24} \right] dy$$

$$= \int_0^1 \left[\frac{y^2+y^4-2y^3}{2} + \frac{(1-y)(1+y^2-2y)}{24} \right] dy$$

$$= \int_0^1 \left[\frac{y^2+y^4-2y^3}{2} + \frac{1+y^2-2y-y-y^3+2y^2}{24} \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[12(y^2+y^4-2y^3) + (1+3y^2-3y-y^3) \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[12y^2+12y^4-24y^3+1+3y^2-3y-y^3 \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[12y^4-25y^3+15y^2-3y+1 \right] dy$$

$$= \frac{1}{24} \left[\frac{12y^5}{5} - 25\frac{y^4}{4} + 15\frac{y^3}{3} - 3\frac{y^2}{2} + y \right]_{y=0}^{1}$$

$$= \frac{1}{24} \left[\left(\frac{12}{5} - \frac{25}{4} + \frac{15}{3} - \frac{3}{2} + 1 \right) - 0 \right]$$

$$= \frac{1}{24} \left[\frac{144-375+300-90+60}{60} \right]$$

$$= \frac{39}{1440}$$

$$= 0.027$$

$$2 \underline{\begin{array}{|l} 5,\, 4,\, 3,\, 2 \\ \hline 5,\, 2,\, 3,\, 1 \end{array}}$$

$$\frac{20 \times 3}{60}$$

3] The joint PDF of the RV is given by,

$$F(x, y) = Kxy \, e^{-(x^2+y^2)}, \quad x>0, y>0.$$

Find i). K ii). check x & y are independent.

Soln.

i). $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y) \, dy \, dx = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} Kxy \, e^{-(x^2+y^2)} \, dy \, dx = 1$$

$$K\int_{0}^{\infty}\int_{0}^{\infty} xy \, e^{-x^2} \, e^{-y^2} \, dy \, dx = 1$$

Take $x^2 = s$ $\quad$ $y^2 = t$

$ds = 2x \, dx$ $\quad$ $2y\,dy = dt$

$\dfrac{ds}{2} = x \, dx$ $\quad$ $y \, dy = \dfrac{dt}{2}$

Now,

$$K\int_{0}^{\infty}\int_{0}^{\infty} e^{-s} e^{-t} \, \frac{dt}{2} \, \frac{ds}{2} = 1$$

$$\frac{K}{4} \int_{0}^{\infty}\int_{0}^{\infty} e^{-s} e^{-t} \, dt \, ds = 1$$

$$\frac{K}{4} \int_{0}^{\infty} e^{-s} \left[\frac{e^{-t}}{-1} \right]_{0}^{\infty} ds = 1$$

$$-\frac{K}{4} \int_{0}^{\infty} e^{-s} [0 - 1] \, ds = 1$$

$$\frac{K}{4} \left(\frac{e^{-s}}{-1} \right)_{0}^{\infty} = 1$$

$$-\frac{K}{4}(0-1) = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

ii). $X \& Y$ are independent.

To prove $F(x,y) = F(x) \cdot f(y)$

Now,

$$F(x) = \int_0^\infty F(x,y)\, dy$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy$$

$$= 4x\, e^{-x^2} \int_0^\infty y\, e^{-y^2}\, dy$$

Take $y^2 = t$

$2y\, dy = dt$

$$y\, dy = \frac{dt}{2}$$

$$= 4x\, e^{-x^2} \int_0^\infty e^{-t}\, \frac{dt}{2}$$

$$= 2x\, e^{-x^2} \left[\frac{e^{-t}}{-1} \right]_0^\infty$$

$$= -2x\, e^{-x^2}(0-1)$$

$$F(x) = 2x\, e^{-x^2}, \quad x > 0$$

$$F(y) = \int_0^\infty F(x,y)\, dx$$

$$= \int_0^\infty 4xy \, e^{-(x^2+y^2)} \, dx$$

$$= 4y \, e^{-y^2} \int_0^\infty x \, e^{-x^2} \, dx$$

Take $x^2 = S$

$2x \, dx = ds$

$x \, dx = \dfrac{ds}{2}$

$$= 4y \, e^{-y^2} \int_0^\infty e^{-S} \, \dfrac{ds}{2}$$

$$= 2y \, e^{-y^2} \left[\dfrac{e^{-S}}{-1} \right]_0^\infty$$

$$= -2y \, e^{-y^2} (0-1)$$

$$F(y) = 2y \, e^{-y^2}, \quad y > 0$$

$$\therefore \ F(x) \cdot F(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$$

$$= 4xy \, e^{-(x^2+y^2)}$$

$$= F(x, y)$$

$$\therefore \ x \ \& \ y \ \text{are independent.}$$

-----

4). If the joint density function of $x \& y$ is given by,

$$F(x, y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

To prove $x \& y$ are independent.

Soln.

WKT $\qquad f(x, y) = \dfrac{\partial^2}{\partial x \, \partial y} F(x, y)$

$$= \frac{\partial^2}{\partial x \partial y} \left[1 - e^{-x} - e^{-y} + e^{-(x+y)} \right]$$

$$= \frac{\partial}{\partial x} \left[-e^{-y}(-1) + e^{-x} e^{-y} (-1) \right]$$

$$= \frac{\partial}{\partial x} \left[e^{-y} - e^{-x} e^{-y} \right]$$

$$= 0 - e^{-y} e^{-x} (-1)$$

$$= e^{-x} e^{-y}$$

$$f(x,y) = e^{-(x+y)}$$

To prove:

$$f(x,y) = f(x) \cdot f(y)$$

Now,

$$f(x) = \int_0^\infty e^{-(x+y)} \, dy$$

$$= e^{-x} \int_0^\infty e^{-y} \, dy$$

$$= e^{-x} \left(\frac{e^{-y}}{-1} \right)_0^\infty$$

$$= -e^{-x} [0 - 1]$$

$$f(x) = e^{-x}$$

$$f(y) = \int_0^\infty e^{-(x+y)} \, dx$$

$$= \int_0^\infty e^{-x} e^{-y} \, dx$$

$$= e^{-y} \left(\frac{e^{-x}}{-1} \right)_0^\infty$$

$$= -e^{-y} (0 - 1)$$

$$f(y) = e^{-y}$$

$$f(x) \cdot f(y) = e^{-x} \cdot e^{-y}$$

$$= e^{-(x+y)}$$

$$= f(x,y)$$

$\therefore$ x and y are independent.