Markov Process:

It is the one in which the future value is independent of the past value given the present value.

Markovian:

A random process $X(t)$ is said to be markovian if

$$P[X(t_{n+1}) \le x_{n+1} / X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots X(t_0) = t_0]$$

$$= P[X(t_{n+1}) \le x_{n+1} / X(t_n) = x_n]$$

where $t_0 \le t_1 \le t_2 \le \ldots \ldots \le t_n \le t_{n+1}$ and

$x_0, x_1, \ldots x_n, x_{n+1}$ are called the states of the process.

Eg:

The probability of raining today depends on the previous whether conditions existed for the last two days and not on the past weather condition.

Markov chain:

If $P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0]$

$= P[X_n = a_n / X_{n-1} = a_{n-1}]$, then the process $X_n$, $n = 0, 1, 2, \ldots$ is called a markov chain.

One-Step Transition Probability:

The conditional probability $P[X_n = a_j / X_{n-1} = a_i]$ is called the one-step transition probability from state $a_i$ to state $a_j$ at the $n^{th}$ step and it is denoted by $P_{ij}(n-1, n)$

Homogeneous Markov chain:

If the one step transition probability does not depend on the step i.e., $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, then the markov chain is called the homogeneous markov chain.

Transition Probability Matrix [TPM]

When the markov chain is homogeneous, the one step transition probability is denoted by $P_{ij}$. The matrix $P_{ij}$ satisfies the following conditions

i). $P_{ij} \geq 0$

ii). $\sum P_{ij} = 1, \ \forall i$

Result:

1]. $P(X_i = a / X_j = b) = P_{ba}^{i-j}$

2]. $P(X_n = j) = \sum_{i=0}^{j} P(X_n = j / X_0 = i) \cdot P(X_0 = i)$

Eg: $P(X_2 = 3 / X_0 = 1) = P_{13}^{2-0} = P_{13}^{(2)}$

J. The tpm of a Markov chain with three states 0, 1, 2 is $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ and the initial state distribution of the chain is $P(X_0 = i) = \frac{1}{3}, i = 0, 1, 2.$

Find i). $P(X_2 = 2)$

ii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

iii). $P(X_2 = 1 / X_0 = 0)$

Soln.

Let $P(X_0 = 0) = \frac{1}{3}$ ; $P(X_0 = 1) = \frac{1}{3}$ ; $P(X_0 = 2) = \frac{1}{3}$

i). $P(X_2 = 2)$

Now, $P(X_n = j) = \sum_{i=0}^{j} P(X_n = j / X_0 = i) \cdot P(X_0 = i)$

$$P(X_2 = 2] = \sum_{i=0}^{2} P(X_2 = 2 / X_0 = i) \; P(X_0 = i)$$

$$= P(X_2 = 2 / X_0 = 0) \; P(X_0 = 0) + P(X_2 = 2 / X_0 = 1) \; P(X_0 = 1) + P(X_2 = 2 / X_0 = 2) \; P(X_0 = 2)$$

$$= P_{02}^{(2-0)} \; P_0^{(0)} + P_{12}^{(2-0)} \; P_1^{(0)} + P_{22}^{(2-0)} \; P_2^{(0)}$$

$$= P_{02}^{(2)} \; P_0^{(0)} + P_{12}^{(2)} \; P_1^{(0)} + P_{22}^{(2)} \; P_2^{(0)} \quad \rightarrow (1)$$

Given $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$

$$P^2 = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.3125 & 0.5 & 0.1875 \\ 0.1875 & 0.5625 & 0.25 \end{pmatrix} \end{array}$$

(i) $\Rightarrow$ $P(X_2 = 2) = 0.0625\left(\frac{1}{3}\right) + 0.1875\left(\frac{1}{3}\right) + 0.25\left(\frac{1}{3}\right)$

$= 0.1639$

ii). $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

$= P(X_3 = 1 / X_2 = 2, X_1 = 1, X_0 = 2)\, P(X_2 = 2, X_1 = 1, X_0 = 2$

$= P(X_3 = 1 / X_2 = 2)\ P(X_2 = 2 / X_1 = 1, X_0 = 2)\ P(X_1 = 1, X_0 = 2)$

$= P(X_3 = 1 / X_2 = 2)\ P(X_2 = 2 / X_1 = 1)\ P(X_1 = 1 / X_0 = 2)\ P(X_0 = 2)$

$= P_{21}^{(1)}\, P_{12}^{(1)}\, P_{21}^{(1)}\, P_2^{(0)}$

$= \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)$

$= 0.046$

iii) $P(X_2 = 1 / X_0 = 0) = P_{01}^{(2)} = 0.31$

Classify the States:

Irreducible:

If for every i, j and for some n such that $P_{ij}^{n} > 0$, then every state can be reached from every other state, then the markov chain is irreducible.

Periodic State:

Let $P_{ij}^{(m)} > 0$ for all m. Let i be a return state.

Then $d_i = GCD\{m : P_{ii}^{(m)} > 0\}$

where GCD stands for the greatest common divisor.

$\Rightarrow$ If $d_i > 1$, then the state 'i' is called periodic

$\Rightarrow$ If $d_i = 1$, then the state 'i' is called aperiodic.

Non-Null Persistant:

If a markov chain is finite and irreducible then all the states are non-null persistant.

Ergodic:

A non-null persistant & aperiodic state is said to be ergodic.

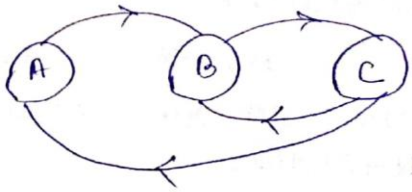
Non Ergodic:

A non-null persistant & periodic state is said to be Non ergodic.

1. Three boys A, B, C are throwing a ball to each other. A always throw a ball to B, B always throw a ball to C, but C is just as likely to throw the ball to B as to A. Find TPM & classify the states, draw diagram.

Soln.

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$




$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

$\therefore\ d_i = GCD\{m, P_{ii}^{(m)} > 0\}$

$d_1 = GCD\{3, 5\} = 1$ $\quad \because P_{11}^{(3)} > 0,\ P_{11}^{(5)} > 0$

$d_2 = GCD\{2, 3, 4, 5\} = 1$

$d_3 = GCD\{2, 3, 4, 5\} = 1$

$\therefore\ d_i = 1 \Rightarrow$ aperiodic

Now, $P_{11}^{(3)} > 0 \quad P_{12}^{(1)} > 0 \quad P_{13}^{(2)} > 0$

$P_{21}^{(2)} > 0 \quad P_{22}^{(2)} > 0 \quad P_{23}^{(1)} > 0$

$P_{31}^{(1)} > 0 \quad P_{32}^{(1)} > 0 \quad P_{33}^{(2)} > 0$

$\therefore$ The chain is irreducible.

Since we have 3 states, the chain is finite and irreducible.

∴ All the states are non-null persistent.
Since all the states are aperiodic & non-null persistent,

∴ It is ergodic.

2) Let $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$, classify the states of the markov chain.

Soln.

Given $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$\therefore P_{11}^{(2)} > 0, \quad P_{22}^{(2)} > 0$ and $P_{33}^{(2)} > 0$

It is irreducible.

Since we've 3 states, the chain is finite and irreducible.
All the states are Non-null Persistant.

$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ ; $P^4 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ ;

$$P^5 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore d_i = GCD\{m, P_{ii}^{(m)} > 0\}$

$d_1 = GCD\{2, 4\} = 2$

$d_2 = GCD\{2, 4\} = 2$

$d_3 = GCD\{2, 4\} = 2$

$\therefore d_i = 2$ It is Periodic

$\therefore$ It is periodic & Non-null persistant.

$\therefore$ It is Non-ergodic

Steady State Distribution :

If $P$ is TPM & Markov chain $\pi = \pi_1 \pi_2 \dots \pi_n$ Steady State Distribution is i). $\pi P = \pi$

ii). $\sum_{i=1}^{n} \pi_i = 1$

1. A house wife buys three kinds of cereals A, B, C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys cereal B. However if she buys either B or C, the next week she is three times as likely to buy A as the other cereal. How often she buys each of the 3 cereals?

Soln.

TPM:

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} \end{array}.$$

Let $\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$

i). $\pi P = \pi$

$$(\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\left(0 + \frac{3\pi_2}{4} + \frac{3\pi_3}{4} \quad \pi_1 + 0 + \frac{\pi_3}{4} \quad 0 + \frac{\pi_2}{4} + 0\right) = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\Rightarrow \frac{3\pi_2 + 3\pi_3}{4} = \pi_1 \; ; \quad \pi_1 + \frac{\pi_3}{4} = \pi_2 \; ; \quad \frac{\pi_2}{4} = \pi_3$$

$$4\pi_1 = 3\pi_2 + 3\pi_3 \; ; \quad 4\pi_1 + \pi_3 = 4\pi_2 \; ; \quad \pi_2 = 4\pi_3$$

$\hookrightarrow (1)$ $\qquad \hookrightarrow (2)$ $\qquad \hookrightarrow (3)$

Subs. (3) in (2),

$$4\pi_1 + \pi_3 - 4(4\pi_3) = 0$$

$$4\pi_1 + \pi_3 - 16\pi_3 = 0$$

$$4\pi_1 - 15\pi_3 = 0$$

$$\pi_1 = \frac{15}{4}\pi_3$$

ii). $\pi_1 + \pi_2 + \pi_3 = 1$

$$\frac{15}{4}\pi_3 + 4\pi_3 + \pi_3 = 1$$

$$\frac{15\pi_3 + 16\pi_3 + 4\pi_3}{4} = 1$$

$$\frac{35\pi_3}{4} = 1$$

$$\pi_3 = \frac{4}{35}$$

$$(3) \Rightarrow \pi_2 = 4\pi_3 = 4\left(\frac{4}{35}\right) = \frac{16}{35} ; \quad \pi_1 = \frac{15}{4}\left(\frac{4}{35}\right) = \frac{15}{35}$$

$$\therefore \pi = \left(\frac{15}{35} \quad \frac{16}{35} \quad \frac{4}{35}\right)$$

---

2]. A man either drives a car or catches a train to his office each day. He never goes two days in row by train, but if he drives one day, then next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day drove to work if and only if a '6' appear. Find

i). the probability that he takes a train on the 3rd day

ii). the probability that he drives to work in the long run?

Soln.

Let T be train and C be car.

Let (T, C) be a travel pattern.

TPM :

$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

prob. of travelling by car = P[getting 6 in the die] $= \frac{1}{6}$

prob. of travelling by train $= 1 - \frac{1}{6} = \frac{5}{6}$

$$\therefore P^{(1)} = \left(\frac{5}{6} \quad \frac{1}{6}\right)$$

$$P^{(2)} = P^{(1)} \cdot P = \left(\frac{5}{6} \quad \frac{1}{6}\right) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{12} \quad \frac{11}{12}\right)$$

$$P^{(3)} = P^{(2)} \cdot P = \left(\frac{1}{12} \quad \frac{11}{12}\right) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{11}{24} \quad \frac{13}{24}\right)$$

i). P[the man travels by train on the 3rd day]

$$= \frac{11}{24}$$

ii). Steady state distribution : $\pi = (\pi_1 \quad \pi_2)$

i). $\pi P = \pi$

$$(\pi_1 \quad \pi_2) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1 \quad \pi_2)$$

$$\left(\frac{\pi_2}{2} \quad \pi_1 + \frac{\pi_2}{2}\right) = (\pi_1 \quad \pi_2)$$

$\Rightarrow \frac{\pi_2}{2} = \pi_1$

$\pi_2 = 2\pi_1$

$\pi_1 + \frac{\pi_2}{2} = \pi_2$

$\Rightarrow 2\pi_1 + \pi_2 = 2\pi_2$

$2\pi_1 = \pi_2$

ii). $\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 + 2\pi_1 = 1 \Rightarrow 3\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{3}$

$\pi_2 = 2/3$

$$\therefore \pi = \left(\frac{1}{3} \quad \frac{2}{3}\right)$$

P[the man travels by car in the long run] $= \frac{2}{3}$