



14) Find the values of a and b so that the surface $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2, -1, -3)$.

Sol.: $a = -\frac{7}{3}$ and $b = \frac{64}{9}$

Divergence of a Vector Point Function:

Let \vec{F} be any given continuously differentiable vector point function then the divergence of \vec{F} is defined as,

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \vec{F} \\ &= \vec{i} \frac{\partial F_x}{\partial x} + \vec{j} \frac{\partial F_y}{\partial y} + \vec{k} \frac{\partial F_z}{\partial z} \end{aligned}$$

Note:

1. $\nabla \cdot \vec{F}$ is a scalar point function.

2. If $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ be a continuously differentiable vector point function then

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal Vectors ::

A vector \vec{F} is said to be solenoidal vector if $\operatorname{div} \vec{F} = 0$.



Curl of a Vector Point function :

Let $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ be any given continuously differentiable Vector Point function, the curl or rotation of \vec{F} is defined as,

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \times \vec{F}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note :: $\nabla \times \vec{F}$ is a Vector Point function.

Irrrotational Vector :

A Vector \vec{F} is said to be irrotational if

$$\nabla \times \vec{F} = 0$$

$$\text{ie.) curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Conservative Vector Field :

If a vector point function \vec{F} is expressible as the gradient of a scalar point function ϕ , then \vec{F} is conservative. ie., \vec{F} is conservative if $\vec{F} = \nabla \phi$. Here ϕ is called scalar potential. \vec{F} is conservative force if $\text{curl } \vec{F} = 0$



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Problems:

1) Prove that $\text{curl}(\nabla\phi) = 0$ (or) $\nabla \times \nabla\phi = 0$

Sol: $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\text{curl}(\nabla\phi) = \nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z} \right) - \vec{j} \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z} \right) + \vec{k} \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y} \right)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\boxed{\text{curl}(\nabla\phi) = 0}$$

2) Prove that $\text{div}(\text{curl } \vec{F}) = 0$ (or) $\nabla \cdot (\nabla \times \vec{F}) = 0$.

Sol: $\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$ if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \vec{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left[\vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \vec{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right]$$



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$$\begin{aligned} \nabla \cdot \nabla \times \vec{F} &= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \\ &\quad + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \end{aligned}$$

= 0

$$\boxed{\nabla \cdot \nabla \times \vec{F} = 0}$$

3) Show that $\text{Curl grad } f = 0$ (or) $\nabla \times \nabla f = 0$

Sol:

$$\text{Curl grad } f = \nabla \times \nabla f$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z} \right] - \vec{j} \left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z} \right]$$

$$+ \vec{k} \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} \right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

= 0

$$\boxed{\text{Curl grad } f = 0}$$

4) If $\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$ what is the directional derivative of v at the point $(1, 2, 3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$.

Sol: $\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$



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$$\nabla v_{(1,2,3)} = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}$$

$$|\vec{a}| = \sqrt{9+16+25} = \sqrt{50}$$

$$\text{Directional derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i} + 3\vec{j} + \vec{k}) \cdot \frac{(3\vec{i} + 4\vec{j} + 5\vec{k})}{\sqrt{50}}$$

$$= \frac{6+12+5}{\sqrt{50}}$$

$$= \frac{23}{\sqrt{50}}$$

$$\text{Directional derivative} = \frac{23}{\sqrt{50}}$$

5) Prove that $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}$

Sol: $\text{div}(\vec{u} \times \vec{v}) = \sum \vec{i} \frac{\partial}{\partial x} (\vec{u} \times \vec{v})$

$$= \sum \vec{i} \left[\vec{u} \times \frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x} \times \vec{v} \right]$$

$$= \sum \vec{i} \cdot \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x} \right) + \sum \vec{i} \cdot \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} \right)$$

$$= \left(\sum \vec{i} \times \frac{\partial \vec{u}}{\partial x} \right) \cdot \vec{v} - \left(\sum \vec{i} \times \frac{\partial \vec{v}}{\partial x} \right) \cdot \vec{u}$$

$$= \text{curl} \vec{u} \cdot \vec{v} - \text{curl} \vec{v} \cdot \vec{u}$$

$$= \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}$$

$$\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl} \vec{u} - \vec{u} \cdot \text{curl} \vec{v}$$



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7) Find the constants a, b, c so that the vector

$$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k} \text{ is}$$

irrotational.

$$\text{Sol: } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (4x+cy+2z) - \frac{\partial}{\partial z} (bx-3y-z) \right]$$

$$- \vec{j} \left[\frac{\partial}{\partial x} (4x+cy+2z) - \frac{\partial}{\partial z} (x+2y+az) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (bx-3y-z) - \frac{\partial}{\partial y} (x+2y+az) \right]$$

$$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$$

Given: \vec{F} is irrotational

$$\text{i.e., } \nabla \times \vec{F} = 0$$

$$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$$

$$c+1=0 \Rightarrow \boxed{c=-1}$$

$$4-a=0 \Rightarrow \boxed{a=4}$$

$$b-2=0 \Rightarrow \boxed{b=2}$$

8) Find "a" so that the vector

$$\vec{A} = (ax^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j} \text{ is irrotational.}$$

Sol: Given: \vec{A} is irrotational.

$$\nabla \times \vec{A} = 0$$



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$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2 - y^2z & -(2xy + y) & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y + 2y)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$\boxed{\nabla \times \vec{A} = 0}$$

$\therefore 'a'$ is arbitrary.

9) Prove $\vec{F} = (y^2 \cos x + z^3) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k}$ is irrotational and find its scalar potential ϕ such that

$$\vec{F} = \nabla \phi.$$

Sol: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix}$

$$= \vec{i} \left[\frac{\partial}{\partial y} (3xz^2) - \frac{\partial}{\partial z} (2y \sin x - 4) \right]$$

$$- \vec{j} \left[\frac{\partial}{\partial x} (3xz^2) - \frac{\partial}{\partial z} (y^2 \cos x + z^3) \right] + \vec{k} \left[\frac{\partial}{\partial x} (2y \sin x - 4) - \frac{\partial}{\partial y} (y^2 \cos x + z^3) \right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.



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To find ϕ :-

$$\nabla\phi = (y^2 \cos x + z^2) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k}$$

We know that $\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = y^2 \cos x + z^2 \Rightarrow \phi = y^2 \sin x + z^2 x + f(y, z)$$

$$\frac{\partial\phi}{\partial y} = 2y \sin x - 4 \Rightarrow \phi = y^2 \sin x - 4y + f(x, z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f(x, y)$$

10) Show that $\vec{F} = (6xy + z^3) \vec{i} + (3x^2 - z) \vec{j} + (3xz^2 - y) \vec{k}$ is irrotational. Find ϕ such that $\vec{F} = \nabla\phi$.

Sol:- $\phi = 3x^2y + xz^3 - yz + c$

11) If $\nabla\phi = yz \vec{i} + xz \vec{j} + xy \vec{k}$ then find ϕ .

Sol:- $\phi = xyz + c$

12) Prove that $\text{div } \hat{r} = \frac{2}{r}$.

Sol:- $\text{div } \hat{r} = \nabla \cdot \left(\frac{\vec{r}}{r} \right)$

$$= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r} \right)$$

$$= \frac{1}{r} - \frac{1}{r^2} \cdot x \frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2} y \frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2} z \frac{\partial r}{\partial z}$$



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$$= \frac{3}{r} - \frac{1}{r^2} \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right]$$

Now, $r^2 = x^2 + y^2 + z^2$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\text{div } \hat{r} = \frac{3}{r} - \frac{1}{r^2} \left[x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r} \right]$

$$= \frac{3}{r} - \frac{1}{r^2} \left[\frac{x^2 + y^2 + z^2}{r} \right]$$

$$= \frac{3}{r} - \frac{1}{r^2} \cdot \frac{r^2}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

$$\boxed{\text{div } \hat{r} = \frac{2}{r}}$$

13) Prove that $(\text{curl curl } \vec{F}) = \nabla(\text{div } \vec{F}) - \nabla^2 \vec{F}$.

Sol: $\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F}) \cdot \nabla - (\nabla \cdot \nabla) \vec{F}$

$$[\therefore \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}]$$

$$= \nabla \cdot (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

$$\boxed{\nabla \times (\nabla \times \vec{F}) = \nabla(\text{div } \vec{F}) - \nabla^2 \vec{F}}$$