



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



14) Prove that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz\vec{k}$ is a conservative force and hence find ϕ so that $\vec{F} = \nabla\phi$.

Sol: \vec{F} is a conservative force i.e., $\text{curl } \vec{F} = 0$

and $\phi = xy^2 + xz^3 + c$ where c is any arbitrary constant.

Green's theorem in a plane:

If R is a closed region of the xy -plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where C is a curve traversed in the anticlockwise direction.

Problems:

1) Evaluate by Green's theorem $\int_C (xy + x^2) dx + (x^2 + y^2) dy$

where C is the square formed by $x = -1, x = 1, y = -1, y = 1$.

Sol: Let R be the region enclosed by C .

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$



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$$\int_c (xy+x^2)dx + (x^2+y^2)dy = \iint_R (2x-x) dx dy$$

$$= \int_{-1}^1 \int_{-1}^1 x dx dy$$

$$= \int_{-1}^1 \left[\frac{x^2}{2} \right]_{-1}^1 dy$$

$$= 0$$

$$\boxed{\int_c (xy+x^2)dx + (x^2+y^2)dy = 0}$$

2) Evaluate by Green's theorem $\int_c e^{-x}(\sin y dx + \cos y dy)$ where c is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$.

Sol: Let R be the region enclosed by c .

By Green's theorem,

$$\int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{Here } M = e^{-x} \sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x} \cos y$$

$$N = e^{-x} \cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x} \cos y$$

$$\int_c e^{-x}(\sin y dx + \cos y dy) = \iint_R (-e^{-x} \cos y - e^{-x} \cos y) dx dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} (-2e^{-x} \cos y) dx dy$$



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$$= -2 \int_0^{\pi/2} \int_0^{\pi} e^{-x} \cos y \, dx \, dy$$
$$= 2(e^{-\pi} - 1)$$

$$\int_c e^{-x} (\sin y \, dx + \cos y \, dy) = 2(e^{-\pi} - 1)$$

3) Evaluate by Green's theorem

$\int_c (x^2 - \cosh y) \, dx + (y + \sin x) \, dy$, where c is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Sol: $\pi(\cosh - 1)$

4) Verify Green's theorem in the plane for

$\int_c (3x^2 - 8y^2) \, dx + (4y - 6xy) \, dy$ where c is the boundary of the region defined by $x = y^2, y = x^2$.

Sol: By Green's theorem,

$$\int_c M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

Given: $\int_c (3x^2 - 8y^2) \, dx + (4y - 6xy) \, dy$

Here $M = 3x^2 - 8y^2 \Rightarrow \frac{\partial M}{\partial y} = -16y$

$N = 4y - 6xy \Rightarrow \frac{\partial N}{\partial x} = -6y$



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Step:1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (-by + by) dx dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 0y dx dy$$

$$= \int_0^1 0y [x]_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 0y (\sqrt{y} - y^2) dy$$

$$= 10 \int_0^1 (y^{3/2} - y^3) dy = 10 \left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4} \right]_0^1$$

$$= 10 \left(\frac{2}{5} - \frac{1}{4} \right) = 10 \left(\frac{8-5}{20} \right)$$

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \frac{3}{2}} \rightarrow \textcircled{1}$$

Step:2

To evaluate

$\int_C M dx + N dy$ we take

C in different Paths

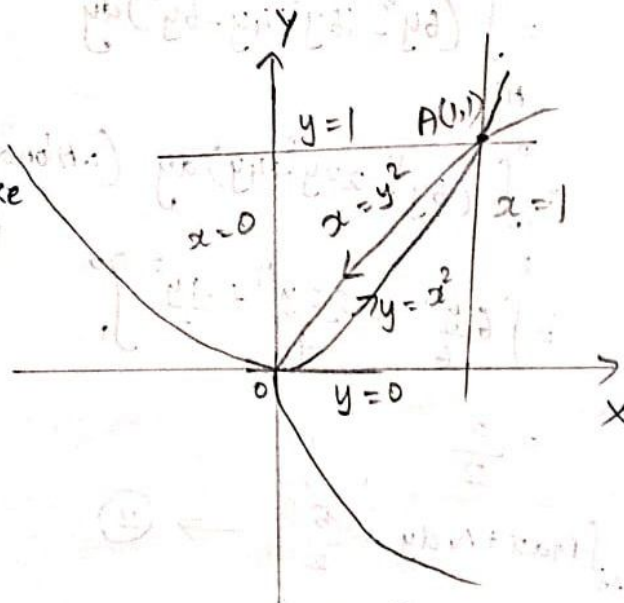
(i) along OA ($y=x^2$)

(ii) along AO ($x=y^2$)

(i) Along OA :-

$$\int_{OA} M dx + N dy = \int_{OA} [3x^2 - 8x^4] dx + [4x^2 - 6x \cdot x^2] 2x dx$$

$(\because x^2 = y, 2x dx = dy)$





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$$\therefore \int_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AO} M dx + N dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$\boxed{\int_C M dx + N dy = \frac{3}{2}} \rightarrow \textcircled{2}$$

$$\textcircled{1} = \textcircled{2}$$

Hence Green's theorem is Verified.

Gauss Divergence theorem:

If \vec{F} is a Vector Point function, finite and differentiable in a region R bounded by a closed surface S , then the surface integral of the normal component of \vec{F} taken over S is equal to the integral of divergence of \vec{F} taken over V .

$$\boxed{\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv}$$

Where \hat{n} is the unit vector in the positive (outward drawn) normal to S .

Problems:

Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.