$$\therefore \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$\boxed{\int_C M\,dx + N\,dy = \frac{3}{2}} \rightarrow (2)$$

$$(1) = (2)$$

Hence Green's theorem is verified.

Gauss Divergence theorem:

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$

where $\hat{n}$ is the unit vector in the positive (outward drawn) normal to S.

Problems:

1) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.

Sol:.

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS:.

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\boxed{\nabla\cdot\vec{F} = 4z - y}$$

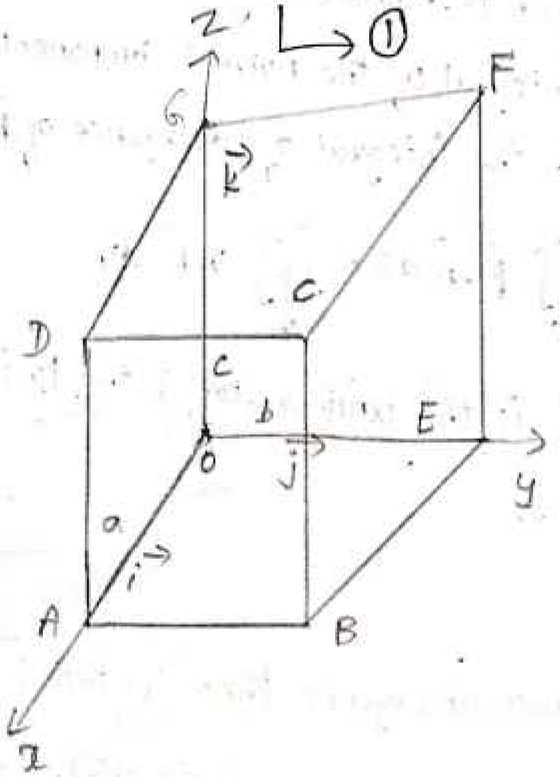
$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz = \frac{3}{2} \quad \rightarrow (1)$$

L.H.S:.

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} +$$

$$\iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	dydz	$x=1$
S_2 - OEFG	$-\vec{i}$	dydz	$x=0$
S_3 - BCEF	$\vec{j}$	dxdz	$y=1$
S_4 - OADG	$-\vec{j}$	dxdz	$y=0$
S_5 - DCGF	$\vec{k}$	dxdy	$z=1$
S_6 - OABE	$-\vec{k}$	dxdy	$z=0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{i}\,dydz$$

$$= \int_0^1\int_0^1 4xz\,dydz$$

$$= \int_0^1\int_0^1 4z\,dydz \qquad (\because x=1)$$

$$= 2$$

$$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 2}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i})\,dydz$$

$$= \int_0^1\int_0^1 (-4xz)\,dydz \qquad (\because x=0)$$

$$= 0$$

$$\boxed{\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{BCEF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 (-y^2)\,dx\,dz \qquad (\text{Here } y=1)$$

$$= \int_0^1\int_0^1 -dx\,dz$$

$$= -1$$

$$\boxed{\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = -1}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{OADG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz = 0 \qquad (\because y=0)$$

$$\boxed{\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DCGF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy \qquad (\because z=1)$$

$$= \int_0^1 y\,[x]_0^1\,dy$$

$$= \int_0^1 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

$$\boxed{\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \frac{1}{2}}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OABE} (-yz)\,dx\,dy = 0 \quad (\because z=0)$$

$$\boxed{\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= 2+0+(-1)+0+\frac{1}{2}+0$$

$$= 2-1+\frac{1}{2}$$

$$= 1+\frac{1}{2} = \frac{2+1}{2} = \frac{3}{2} \rightarrow (2)$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}}$$

From (1) & (2), $\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$

2) Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - yx)\vec{k}$ taken over the rectangle Parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

(or)

Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the Surface of the rectangular Parallelopiped bounded by $x=0, x=a, y=0, y=b$ $z=0, z=c$.

Sol: By Gauss-divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS :-

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\boxed{\nabla\cdot\vec{F} = 2(x+y+z)}$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a\int_0^b \left[xz+yz+\frac{z^2}{2}\right]_0^c dx\,dy$$

$$= 2\int_0^a\int_0^b \left(xc+yc+\frac{c^2}{2}\right)dx\,dy$$

$$= 2\int_0^a \left[xyc+\frac{y^2}{2}c+\frac{c^2}{2}y\right]_0^b dx$$

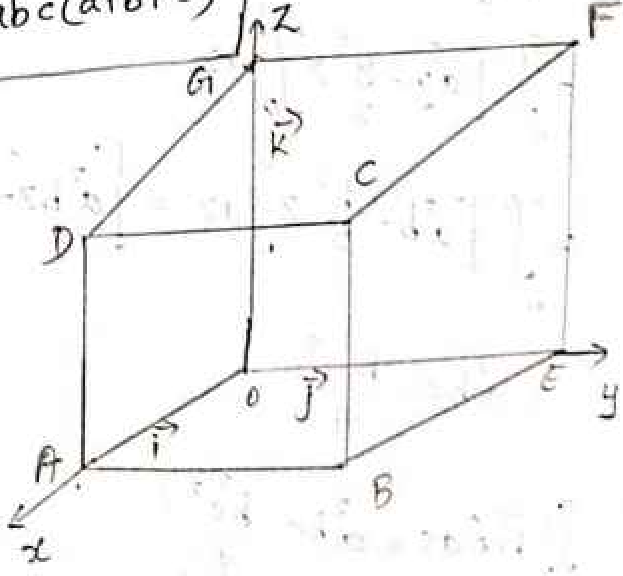
$$= 2\int_0^a \left(xbc+\frac{b^2c}{2}+\frac{c^2b}{2}\right)dx$$

$$= 2\left[\frac{x^2}{2}bc+\frac{b^2c}{2}x+\frac{c^2b}{2}x\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= abc(a+b+c) \longrightarrow (1)$$

$$\boxed{\iiint_V \nabla\cdot\vec{F}\,dV = abc(a+b+c)}$$



LHS :

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	dydz	$x=a$
S_2 - OEFG	$-\vec{i}$	dydz	$x=0$
S_3 - BCEF	$\vec{j}$	dxdz	$y=b$
S_4 - OADG	$-\vec{j}$	dxdz	$y=0$
S_5 - DCGF	$\vec{k}$	dxdy	$z=c$
S_6 - OABE	$-\vec{k}$	dxdy	$z=0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} \left\{ [x^2-yz]\vec{i} + [y^2-zx]\vec{j} + (z^2-xy)\vec{k} \right\} \cdot \vec{i}\,dy\,dz$$

$$= \int_0^c \int_0^b (x^2-yz)\,dy\,dz$$

$$= \int_0^c \int_0^b (a^2-yz)\,dy\,dz \qquad (\because x=a)$$

$$= \int_0^c \left[a^2y - \frac{y^2}{2}z\right]_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2}{2}z\right] dz = \left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4}}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} \left[(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\right](-i)\,dy\,dz$$

$$= \int_0^c\int_0^b -(x^2-yz)\,dy\,dz$$

$$= \int_0^c\int_0^b yz\,dy\,dz \qquad (\because x=0)$$

$$= \int_0^c \left[\frac{y^2}{2}\right]_0^b z\,dz$$

$$= \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2c^2}{4}$$

$$\boxed{\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4}}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{BCEF} \left[(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\right]\cdot\vec{j}\,dx\,dz$$

$$= \int_0^a\int_0^c (y^2-zx)\,dx\,dz$$

$$= \int_0^a\int_0^c (b^2-zx)\,dx\,dz \qquad (\because y=b)$$

$$= \int_0^a \left[b^2x-\frac{zx^2}{2}\right]_0^c dz$$

$$= \int_0^a \left[b^2c-\frac{zc^2}{2}\right]dz$$

$$= \left[b^2 c z - \frac{z^2}{4} c^2\right]_0^a$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\boxed{\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = ab^2c - \frac{a^2c^2}{4}}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint_{OADG} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right] (-\vec{j})\, dx\, dz$$

$$= \int_0^a\int_0^c -(y^2-zx)\, dx\, dz$$

$$= \int_0^a\int_0^c zx\, dx\, dz \qquad (\because y=0)$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^c z\, dz$$

$$= \frac{c^2}{2}\left[\frac{z^2}{2}\right]_0^a$$

$$= \frac{a^2c^2}{2}$$

$$\boxed{\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \frac{a^2c^2}{2}}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint_{DCGF} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right] \vec{k}\, dx\, dy$$

$$= \int_0^a \int_0^b (z^2 - xy)\,dx\,dy \qquad (\because z = c)$$

$$= \int_0^a \int_0^b (c^2 - xy)\,dx\,dy$$

$$= \int_0^a \left[c^2 x - \frac{x^2}{2} y\right]_0^b dy$$

$$= \int_0^a \left(bc^2 - \frac{b^2 y}{2}\right) dy$$

$$= \left(bc^2 y - \frac{b^2 y^3}{2}\right)_0^a$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

$$\boxed{\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = abc^2 - \frac{a^2 b^2}{4}}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OABE} \left[(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right].(-\vec{k})\,dx\,dy$$

$$= \int_0^a \int_0^b -(z^2 - xy)\,dx\,dy \qquad (\because z = 0)$$

$$= \int_0^a \int_0^b xy\,dx\,dy$$

$$= \int_0^a y\,dy \left[\frac{x^2}{2}\right]_0^b = \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^2 b^2}{4}$$

$$\boxed{\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \frac{a^2 b^2}{4}}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{a^2c^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= abc(a+b+c) \rightarrow (2)$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)}$$

From (1) and (2),

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

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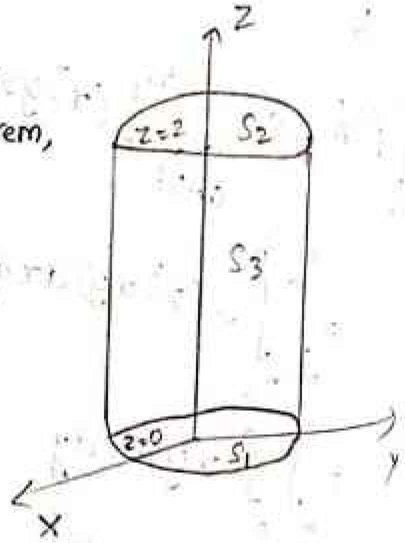
3) Verify Gauss divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$.



Sol: By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS:

Given: $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(y\vec{i} + x\vec{j} + z^2\vec{k})$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$$

$$= 0 + 0 + 2z$$