



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)



Stoke's theorem:

If \vec{F} is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

where \hat{n} is the unit normal vector at any point of S .

Note:

1. If \vec{F} is irrotational, $\nabla \times \vec{F} = 0$

$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 0$ and hence \vec{F} is conservative.

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P dx + Q dy + R dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Problems:

1) Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.

Sol: By Stoke's theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

RHS:-

Given: $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$



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$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y)$$

$$= 4y\vec{k}$$

$$\boxed{\nabla \times \vec{F} = 4y\vec{k}}$$

Here the Surface S denotes the rectangle $OABC$ and the unit outward normal vector is \vec{k} .

$$\text{i.e., } \hat{n} = \vec{k}$$

$$\text{Curl } \vec{F} \cdot \hat{n} \, ds = 4y\vec{k} \cdot \vec{k} \, dx \, dy \\ = 4y \, dx \, dy$$

$$\therefore \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = \iint_S 4y \, dx \, dy$$

$$= \int_0^b \int_0^a 4y \, dx \, dy$$

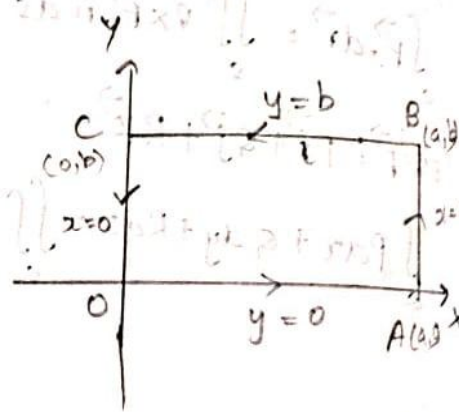
$$= 4 \int_0^b [x]_0^a y \, dy$$

$$= 4a \left[\frac{y^2}{2} \right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \rightarrow \textcircled{1}$$

$$\boxed{\iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 2ab^2}$$





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L.H.S

$$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xy dy$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [(x^2 - y^2)dx + 2xy dy]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA ($y=0$):

$$\int_{OA} (x^2 - y^2) dx + 2xy dy = \int_0^a x^2 dx$$

$$[\because y=0 \Rightarrow dy=0$$

Along OA, $x \rightarrow 0$ to a]

$$= \left[\frac{x^3}{3} \right]_0^a$$

$$= \frac{a^3}{3}$$

$$\int_{OA} (x^2 - y^2) dx + 2xy dy = \frac{a^3}{3}$$

Along AB ($x=a$):

$$\int_{AB} (x^2 - y^2) dx + 2xy dy = \int_0^b 2ay dy$$

$$[\because x=a \Rightarrow dx=0$$

Along AB, $y \rightarrow 0$ to b]

$$= 2a \left[\frac{y^2}{2} \right]_0^b$$

$$= 2a \left[\frac{b^2}{2} \right]$$

$$\int_{AB} (x^2 - y^2) dx + 2xy dy = ab^2$$



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Along Bc ($y=b$) \therefore

$$\int_{Bc} (x^2 - y^2) dx + 2xy dy = \int_a^0 (x^2 - b^2) dx$$

$$= \left[\frac{x^3}{3} - b^2 x \right]_a^0$$
$$= -\frac{a^3}{3} + ab^2$$

$$\therefore [y=b \Rightarrow dy=0]$$

Along Bc $x \rightarrow a \rightarrow 0$

$$\int_{Bc} (x^2 - y^2) dx + 2xy dy = -\frac{a^3}{3} + ab^2$$

Along Co ($x=0$) \therefore

$$\int_{Co} (x^2 - y^2) dx + 2xy dy = \int_{Co} 0 + 0 = 0 \quad (\because x=0, dx=0)$$

$$\int_{Co} (x^2 - y^2) dx + 2xy dy = 0$$

$$\text{Hence } \int_C \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{Bc} + \int_{Co}$$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$
$$= ab^2 + ab^2$$

$$\int_C \vec{F} \cdot d\vec{r} = 2ab^2 \rightarrow (2)$$

$$\text{From (1) \& (2), } \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} ds$$

Hence Stoke's theorem is verified.



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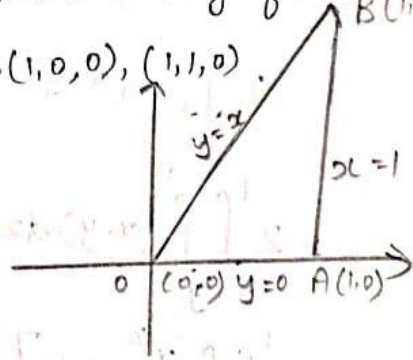


2) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by Stokes's theorem where

$\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$ and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

Sol.: By Stokes's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F} = y^2 \vec{i} + x^2 \vec{j} - (x+z) \vec{k}$$
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y) \vec{k}$$

$$\boxed{\nabla \times \vec{F} = \vec{j} + 2(x-y) \vec{k}}$$

Since the triangle lies on xy plane and hence the unit vector normal to the surface OAB is \vec{k} .

$$\text{i.e., } \hat{n} = \vec{k}$$

$$\nabla \times \vec{F} \cdot \hat{n} = [\vec{j} + 2(x-y) \vec{k}] \cdot \vec{k} = 2(x-y)$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



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$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

$$= \iint_S 2(x-y) \, dx \, dy \quad (\because S \text{ lies on } xy \text{ plane} \\ ds = dx \, dy)$$

$$= 2 \int_0^1 \int_y^1 (x-y) \, dx \, dy$$

$$= 2 \int_0^1 \left[\frac{x^2}{2} - xy \right]_y^1 \, dy$$

$$= 2 \int_0^1 \left[\frac{1}{2} - y - \frac{y^2}{2} + y^2 \right] \, dy$$

$$= 2 \left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3} \right]_0^1$$

$$= 2 \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3} \right)$$

$$= 2 \left(-\frac{1}{6} + \frac{1}{3} \right)$$

$$= 2 \left(\frac{-1+2}{6} \right)$$

$$= 2 \left(\frac{1}{6} \right)$$

$$= \frac{1}{3}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \frac{1}{3}}$$