Divergence of a Vector Point Function:

Let $\vec{F}$ be any given continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as,

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$= \vec{i}\,\frac{\partial \vec{F}}{\partial x} + \vec{j}\,\frac{\partial \vec{F}}{\partial y} + \vec{k}\,\frac{\partial \vec{F}}{\partial z}$$

Note:

1. $\nabla \cdot \vec{F}$ is a scalar point function.
2. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuously differentiable vector point function then

$$\boxed{\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Curl of a Vector Point function:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl or rotation of $\vec{F}$ is defined as,

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: $\nabla \times \vec{F}$ is a vector point function.

Problems:-

1) Prove that $\text{curl}(\nabla\phi) = 0$ (or) $\nabla \times \nabla\phi = 0$

Sol:-

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{curl}(\nabla\phi) = \nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\boxed{\text{curl}(\nabla\phi) = 0}$$

2) Prove that $\text{div}(\text{curl}\,\vec{F}) = 0$ (or) $\nabla\cdot(\nabla\times\vec{F}) = 0$

Sol:-

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \text{if } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\nabla\times\vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla\cdot\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$\nabla\cdot\nabla\times\vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= 0$$

$$\boxed{\nabla\cdot\nabla\times\vec{F} = 0}$$

3) Show that Curl grad $f = 0$ (or) $\nabla\times\nabla f = 0$

Sol:. Curl grad $f = \nabla\times\nabla f$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right] - \vec{j}\left[\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right] + \vec{k}\left[\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$\boxed{\text{Curl grad } f = 0}$$

4) If $\nabla V = y\vec{i} + z\vec{j} + x\vec{k}$ what is the directional derivative of V at the Point $(1,2,3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$.

Sol:. $\nabla V = y\vec{i} + z\vec{j} + x\vec{k}$

$$\nabla v_{(1,2,3)} = 2\vec{i}+3\vec{j}+\vec{k}$$

$$\vec{a} = 3\vec{i}+4\vec{j}+5\vec{k}$$

$$|\vec{a}| = \sqrt{9+16+25} = \sqrt{50}$$

$$\text{Directional derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i}+3\vec{j}+\vec{k}) \cdot \frac{(3\vec{i}+4\vec{j}+5\vec{k})}{\sqrt{50}}$$

$$= \frac{6+12+5}{\sqrt{50}}$$

$$= \frac{23}{\sqrt{50}}$$

$$\boxed{\text{Directional derivative} = \frac{23}{\sqrt{50}}}$$

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5) Prove that $\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{curl}\vec{u} - \vec{u}\operatorname{curl}\vec{v}$

Sol:

$$\operatorname{div}(\vec{u}\times\vec{v}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \sum \vec{i}\left[\vec{u}\times\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\times\vec{v}\right]$$

$$= \sum \vec{i}\cdot\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right) + \sum \vec{i}\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right)$$

$$= \left(\sum \vec{i}\times\frac{\partial\vec{u}}{\partial x}\right)\cdot\vec{v} - \left(\sum \vec{i}\times\frac{\partial\vec{v}}{\partial x}\right)\cdot\vec{u}$$

$$= \operatorname{curl}\vec{u}\cdot\vec{v} - \operatorname{curl}\vec{v}\cdot\vec{u}$$

$$= \vec{v}\operatorname{curl}\vec{u} - \vec{u}\operatorname{curl}\vec{v}$$

$$\boxed{\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}}$$
