Solenoidal Vector :

A Vector $\vec{F}$ is said to be Solenoidal Vector if $\text{div}\,\vec{F} = 0$.

Irrotational Vector :

A Vector $\vec{F}$ is said to be irrotational if

$$\nabla \times \vec{F} = 0$$

$$\text{i.e.)}\ \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Conservative Vector Field :

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function ϕ, then $\vec{F}$ is conservative. ie.,) $\vec{F}$ is conservative if $\vec{F} = \nabla\phi$. Here ϕ is called scalar potential.

$\vec{F}$ is Conservative force if $\text{curl}\,\vec{F} = 0$

6) Prove that $r^n \vec{r}$ is solenoidal if $n=-3$ and $r^n \vec{r}$ is irrotational for all values of n.

Soln:

$$r^n \vec{r} = r^n(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\operatorname{div}(r^n \vec{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

Now $r^2 = x^2+y^2+z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\frac{\partial}{\partial x}(r^n x) = x\frac{\partial}{\partial r}(r^n)\frac{\partial r}{\partial x} + r^n$

$$= xnr^{n-1}\frac{x}{r} + r^n$$

$$= x^2 n r^{n-2} + r^n$$

$$\boxed{\frac{\partial}{\partial x}(r^n x) = x^2 n r^{n-2} + r^n}$$

$$\frac{\partial}{\partial y}(r^n y) = y^2 n r^{n-2} + r^n$$

$$\frac{\partial}{\partial z}(r^n z) = z^2 n r^{n-2} + r^n$$

$$\text{div}(r^n \vec{r}) = (x^2+y^2+z^2)\, n r^{n-2} + 3r^n$$

$$= r^2 n r^{n-2} + 3r^n$$

$$= n r^n + 3r^n$$

$$= (n+3) r^n$$

$$\boxed{\text{div}(r^n \vec{r}) = (n+3) r^n}$$

The Vector $r^n \vec{r}$ is solenoidal if,

$$\text{div}(r^n \vec{r}) = 0$$

$$(n+3) r^n = 0$$

$$n+3 = 0$$

$$\boxed{n = -3}$$

$\therefore r^n \vec{r}$ is solenoidal only if $n = -3$.

$$\text{Curl}(r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \vec{i}\, \frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)$$

$$= \sum \vec{i} \left\{ n r^{n-1} \frac{\partial r}{\partial y} z - n r^{n-1} \frac{\partial r}{\partial z} \cdot y \right\}$$

$$= \sum \vec{i} \left\{ n r^{n-1} \left(\frac{yz}{r} - \frac{zy}{r} \right) \right\} = 0$$

$$\boxed{\text{Curl}(r^n \vec{r}) = \sum \vec{i} \left\{ n r^{n-1} \left(\frac{yz}{r} - \frac{zy}{r} \right) \right\}}$$

7) Find the constants a, b, c so that the vector

$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Sol: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= \vec{i}\left[\frac{\partial}{\partial y}[4x+cy+2z] - \frac{\partial}{\partial z}(bx-3y-z)\right]$

$-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$

$+\vec{k}\left[\frac{\partial}{\partial x}[bx-3y-z] - \frac{\partial}{\partial y}(x+2y+az)\right]$

$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$

Given : $\vec{F}$ is irrotational

i.e.,) $\nabla \times \vec{F} = 0$

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$

$c+1 = 0 \Rightarrow \boxed{c=-1}$

$4-a = 0 \Rightarrow \boxed{a=4}$

$b-2 = 0 \Rightarrow \boxed{b=2}$

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8) Find "a" so that the vector

$\vec{A} = (ax^2-y^2+x)\vec{i} - (2xy+y)\vec{j}$ is irrotational.

Sol: Given : $\vec{A}$ is irrotational.

$\nabla \times \vec{A} = 0$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2-y^2+x & -(2xy+y) & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y+2y)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$\boxed{\nabla \times \vec{A} = 0}$$

$\therefore$ 'a' is arbitrary.

9) Prove $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar Potential ϕ such that $\vec{F} = \nabla\phi$.

Sol:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

To find ϕ :-

$$\nabla\phi = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

We know that $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi = y^2\sin x + z^3x + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi = y^2\sin x - 4y + f(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f(x,y)$$

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10) Show that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational. Find ϕ such that $\vec{F} = \nabla\phi$.

Sol:- $\phi = 3x^2y + xz^3 - yz + c$

11) If $\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$ then find ϕ.

Sol:- $\phi = xyz + c$

12) Prove that $\operatorname{div}\hat{r} = \dfrac{2}{r}$.

Sol:- $\operatorname{div}\hat{r} = \nabla\cdot\left(\dfrac{\vec{r}}{r}\right)$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r} - \frac{1}{r^2}\cdot x\frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2}y\frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2}z\frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{1}{r^2}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Now, $r^2 = x^2+y^2+z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\operatorname{div}\hat{r} = \frac{3}{r} - \frac{1}{r^2}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$

$$= \frac{3}{r} - \frac{1}{r^2}\left[\frac{x^2+y^2+z^2}{r}\right]$$

$$= \frac{3}{r} - \frac{1}{r^2}\cdot\frac{r^2}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

$$\boxed{\operatorname{div}\hat{r} = \frac{2}{r}}$$

13) Prove that $(\text{curl curl } \vec{F}) = \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}$.

Sol: $\nabla\times(\nabla\times\vec{F}) = (\nabla\cdot\vec{F})\cdot\nabla - (\nabla\cdot\nabla)\vec{F}$

$$\left[\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}\right]$$

$$= \nabla\cdot(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

$$\boxed{\nabla\times(\nabla\times\vec{F}) = \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}}$$