Green's theorem in a plane:

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\boxed{\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where C is a curve traversed in the anticlockwise direction.

Problems:

1) Evaluate by Green's theorem $\int_C (xy + x^2)\,dx + (x^2 + y^2)\,dy$ where C is the square formed by $x = -1, x = 1, y = -1, y = 1$.

Sol: Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

$$\int_C (xy+x^2)dx + (x^2+y^2)dy = \iint_R (2x-x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1}\left[\frac{x^2}{2}\right]_{-1}^{1} dy$$

$$= 0$$

$$\boxed{\int_C (xy+x^2)dx + (x^2+y^2)dy = 0}$$

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2) Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where c is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$.

Sol:. Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Here $M = e^{-x}\sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

$$\int_C e^{-x}(\sin y\,dx + \cos y\,dy) = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-2e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_0^{\pi/2}\int_0^{\pi} e^{-x}\cos y \, dx\, dy$$

$$= 2(e^{-\pi}-1)$$

$$\boxed{\int_C e^{-x}(\sin y\, dx + \cos y\, dy) = 2(e^{-\pi}-1)}$$

3) Evaluate by Green's theorem $\int_C (x^2-\cosh y)dx + (y+\sin x)dx$, where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Sol: $\pi(\cosh - 1)$