4) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where $C$ is the boundary of the region defined by $x=y^2$, $y=x^2$.

Sol: By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Given: $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

Here $M = 3x^2-8y^2 \Rightarrow \dfrac{\partial M}{\partial y} = -16y$

$N = 4y-6xy \Rightarrow \dfrac{\partial N}{\partial x} = -6y$

Step:1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\,dx\,dy$$

$$= \int_0^1 10y \left[x\right]_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 10y\,(\sqrt{y} - y^2)\,dy$$

$$= 10\int_0^1 (y^{3/2} - y^3)\,dy = 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left(\frac{2}{5} - \frac{1}{4}\right) = 10\left(\frac{8-5}{20}\right)$$

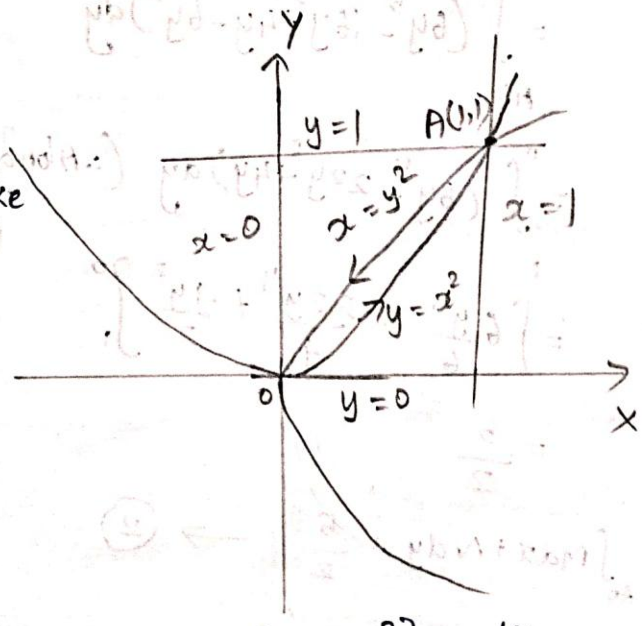
$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{2}} \longrightarrow (1)$$

Step:2

To evaluate $\int_C M\,dx + N\,dy$ we take C in different Paths

(i) along OA $(y = x^2)$

(ii) along AO $(x = y^2)$




(i) Along OA :-

$$\int_{OA} M\,dx + N\,dy = \int_{OA} [3x^2 - 8x^4]\,dx + [4x^2 - 6x \cdot x^2]\,2x\,dx$$

$$(\because x^2 = y,\ 2x\,dx = dy)$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx \quad (\because \text{Along } OA,\ x \text{ varies from } 0 \text{ to } 1)$$

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2)\,dx$$

$$= \left[-20\frac{x^5}{5} + \frac{8x^4}{4} + \frac{3x^3}{3}\right]_0^1$$

$$= -1$$

$$\boxed{\int_{OA} M\,dx + N\,dy = -1}$$

ii) Along $AO$:

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (3y^4 - 8y^2)\,2y\,dy + (4y - 6yy)\,dy$$

$$(\because y^2 = x,\ 2y\,dy = dx)$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy \quad (\because \text{Along } AO\ y \text{ varies from } 1 \text{ to } 0)$$

$$= \left[6\frac{y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= \frac{5}{2}$$

$$\boxed{\int_{AO} M\,dx + N\,dy = \frac{5}{2}} \rightarrow (2)$$

$$\therefore \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$\boxed{\int_C M\,dx + N\,dy = \frac{3}{2}} \rightarrow (2)$$

$$(1) = (2)$$

Hence Green's theorem is verified.