Gauss Divergence theorem:

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$

where $\hat{n}$ is the unit vector in the positive (outward drawn) normal to S.

Problems:

1) Verify Gauss divergence theorem for $\vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.

Sol:-

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS:-

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\cdot(4xz\vec{i}-y^2\vec{j}+yz\vec{k})$$

$$= \frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}(-y^2)+\frac{\partial}{\partial z}(yz)$$

$$= 4z-2y+y$$

$$= 4z-y$$

$$\boxed{\nabla\cdot\vec{F} = 4z-y}$$

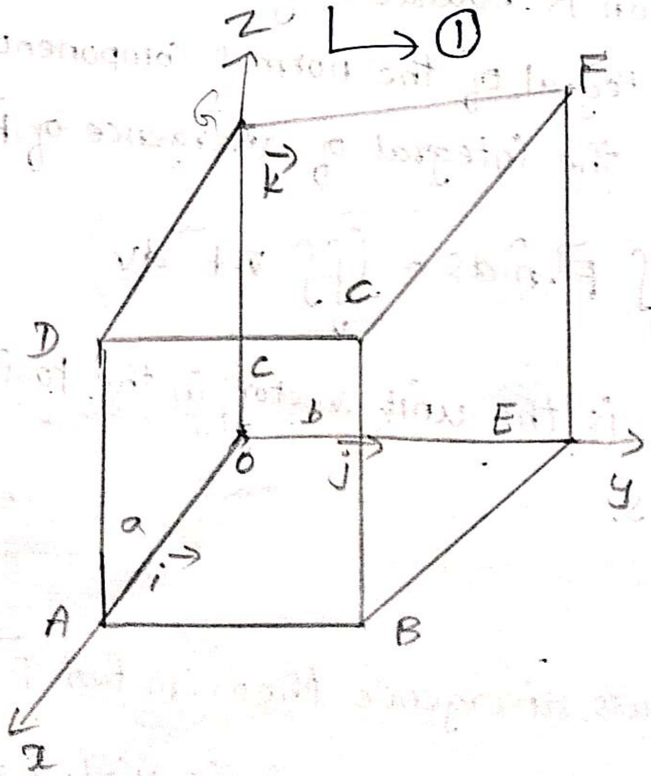
$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz = \frac{3}{2} \quad \rightarrow (1)$$

L.H.S:-

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} +$$

$$\iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$



Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	$dydz$	$x=1$
S_2 - OEFG	$-\vec{i}$	$dydz$	$x=0$
S_3 - BCEF	$\vec{j}$	$dxdz$	$y=1$
S_4 - OADG	$-\vec{j}$	$dxdz$	$y=0$
S_5 - DCGF	$\vec{k}$	$dxdy$	$z=1$
S_6 - OABE	$-\vec{k}$	$dxdy$	$z=0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} (4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot\vec{i}\,dydz$$

$$= \int_0^1\int_0^1 4xz\,dydz$$

$$= \int_0^1\int_0^1 4z\,dydz \qquad (\because x=1)$$

$$= 2$$

$$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 2}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} (4xz\vec{i}-y^2\vec{j}+yz\vec{k})(-\vec{i})\,dydz$$

$$= \int_0^1\int_0^1 (-4xz)\,dydz \qquad (\because x=0)$$

$$= 0$$

$$\boxed{\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{BCEF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 (-y^2)\,dx\,dz \qquad (\text{Here } y=1)$$

$$= \int_0^1\int_0^1 -dx\,dz$$

$$= -1$$

$$\boxed{\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = -1}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{OADG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz = 0 \qquad (\because y=0)$$

$$\boxed{\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DCGF} (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy \qquad (\because z=1)$$

$$= \int_0^1 y\,[x]_0^1\,dy$$

$$= \int_0^1 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

$$\boxed{\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \frac{1}{2}}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OABE} (-yz)\,dx\,dy = 0 \quad (\because z=0)$$

$$\boxed{\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= 2 + 0 + (-1) + 0 + \frac{1}{2} + 0$$

$$= 2 - 1 + \frac{1}{2}$$

$$= 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2} \longrightarrow (2)$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}}$$

From (1) & (2), $\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$