2) Verify divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-yx)\vec{k}$ taken over the rectangle Parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

(or)

Verify Gauss divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ and S is the Surface of the rectangular Parallelopiped bounded by $x=0, x=a, y=0, y=b$ $z=0, z=c.$

Sol:- By Gauss-divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS :-

$$\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left((x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x+2y+2z$$

$$= 2(x+y+z)$$

$$\boxed{\nabla\cdot\vec{F} = 2(x+y+z)}$$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a\int_0^b \left[xz+yz+\frac{z^2}{2}\right]_0^c dx\,dy$$

$$= 2\int_0^a\int_0^b \left(xc+yc+\frac{c^2}{2}\right)dx\,dy$$

$$= 2\int_0^a \left[xyc+\frac{y^2}{2}c+\frac{c^2}{2}y\right]_0^b dx$$

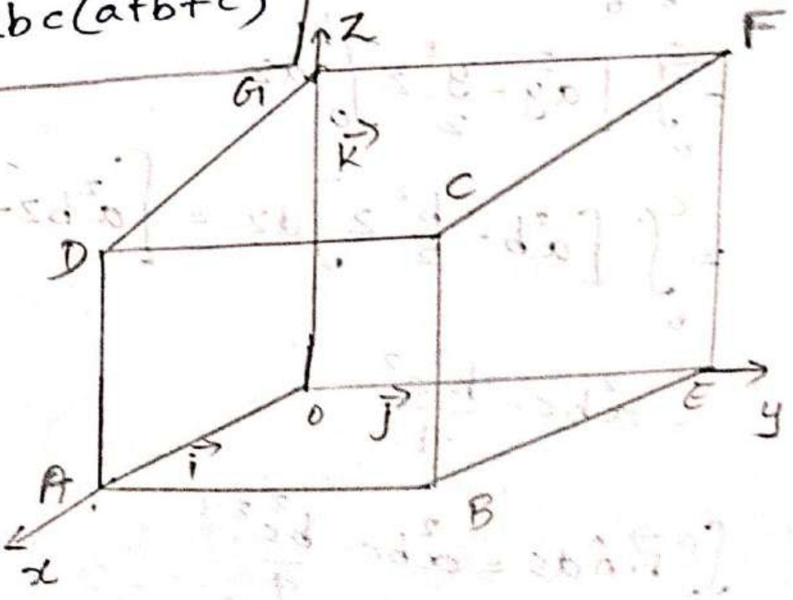
$$= 2\int_0^a \left(xbc+\frac{b^2c}{2}+\frac{c^2b}{2}\right)dx$$

$$= 2\left[\frac{x^2}{2}bc+\frac{b^2c}{2}x+\frac{c^2b}{2}x\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$$= abc(a+b+c) \longrightarrow (1)$$

$$\boxed{\iiint_V \nabla\cdot\vec{F}\,dV = abc(a+b+c)}$$



LHS:

$$\iint_S \vec{F}\cdot\hat{n}\,ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	$\hat{n}$	ds	Face equation
S_1 - ABCD	$\vec{i}$	$dydz$	$x=a$
S_2 - OEFG	$-\vec{i}$	$dydz$	$x=0$
S_3 - BCEF	$\vec{j}$	$dxdz$	$y=b$
S_4 - OADG	$-\vec{j}$	$dxdz$	$y=0$
S_5 - DCGF	$\vec{k}$	$dxdy$	$z=c$
S_6 - OABE	$-\vec{k}$	$dxdy$	$z=0$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{ABCD} \{[x^2-yz]\vec{i} + [y^2-zx]\vec{j} + (z^2-xy)\vec{k}\}\cdot\vec{i}\,dy\,dz$$

$$= \int_0^c\int_0^b (x^2-yz)\,dy\,dz$$

$$= \int_0^c\int_0^b (a^2-yz)\,dy\,dz \qquad (\because x=a)$$

$$= \int_0^c \left[a^2y - \frac{y^2}{2}z\right]_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2}{2}z\right]dz = \left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4}}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{OEFG} \left[(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\right](-i)\,dy\,dz$$

$$= \int_0^c\int_0^b -(x^2-yz)\,dy\,dz$$

$$= \int_0^c\int_0^b yz\,dy\,dz \qquad (\because x=0)$$

$$= \int_0^c \left[\frac{y^2}{2}\right]_0^b z\,dz$$

$$= \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2c^2}{4}$$

$$\boxed{\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4}}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{BCEF} \left[(x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\right]\cdot\vec{j}\,dx\,dz$$

$$= \int_0^a\int_0^c (y^2-zx)\,dx\,dz$$

$$= \int_0^a\int_0^c (b^2-zx)\,dx\,dz \qquad (\because y=b)$$

$$= \int_0^a \left[b^2x-\frac{zx^2}{2}\right]_0^c dz$$

$$= \int_0^a \left[b^2c-\frac{zc^2}{2}\right]dz$$

$$= \left[b^2cz - \frac{z^2}{4}c^2\right]_0^a$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\boxed{\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = ab^2c - \frac{a^2c^2}{4}}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{OADG} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right] (-\vec{j})\,dx\,dz$$

$$= \int_0^a\int_0^c -(y^2-zx)\,dx\,dz$$

$$= \int_0^a\int_0^c zx\,dx\,dz \qquad (\because y=0)$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^c z\,dz$$

$$= \frac{c^2}{2}\left[\frac{z^2}{2}\right]_0^a$$

$$= \frac{a^2c^2}{2}$$

$$\boxed{\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \frac{a^2c^2}{2}}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \iint_{DCGF} \left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right] \vec{k}\,dx\,dy$$

$$= \int_0^a \int_0^b (z^2 - xy)\,dx\,dy$$

$$= \int_0^a \int_0^b (c^2 - xy)\,dx\,dy \qquad (\because z = c)$$

$$= \int_0^a \left[c^2 x - \frac{x^2}{2} y\right]_0^b dy$$

$$= \int_0^a \left(bc^2 - \frac{b^2 y}{2}\right) dy$$

$$= \left(bc^2 y - \frac{b^2 y^2}{2}\right)_0^a$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

$$\boxed{\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = abc^2 - \frac{a^2 b^2}{4}}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{OABE} \left[(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right]\cdot(-\vec{k})\,dx\,dy$$

$$= \int_0^a \int_0^b -(z^2 - xy)\,dx\,dy$$

$$= \int_0^a \int_0^b xy\,dx\,dy \qquad (\because z = 0)$$

$$= \int_0^a y\,dy \left[\frac{x^2}{2}\right]_0^b = \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^2 b^2}{4}$$

$$\boxed{\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \frac{a^2 b^2}{4}}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{a^2c^2}{4} + \frac{a^2c^2}{4} + ab_{\text{,}}$$
$$- \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= abc(a+b+c) \longrightarrow ②$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)}$$

From ① and ②,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

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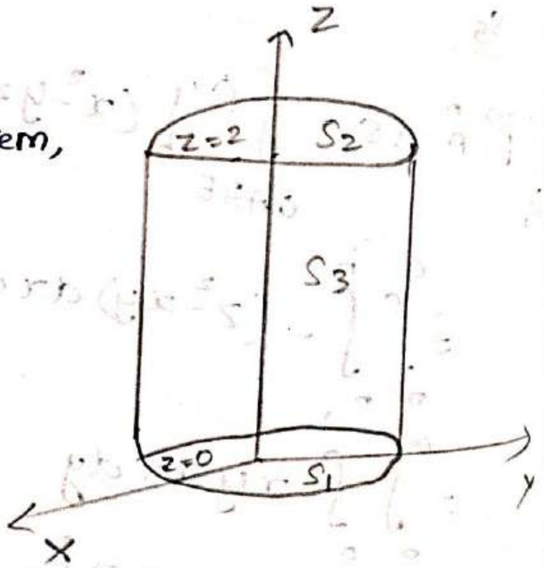
3) Verify Gauss divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the Cylindrical region bounded by $x^2+y^2 = 9$, $z=0$ and $z=2$.



Sol: By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS:

Given: $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(y\vec{i} + x\vec{j} + z^2\vec{k})$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$$

$$= 0 + 0 + 2z$$

$= 2z$

$\nabla \cdot \vec{F} = 2z$

The region is bounded by $z=0$ and $z=2$

$x^2+y^2=9$

$y^2 = 9-x^2$

$y = \pm\sqrt{9-x^2}$

$$\therefore \iiint_V \nabla \cdot \vec{F}\, dv = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{2} 2z\,dz\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left(\frac{z^2}{2}\right)_0^2 dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (z^2)^2\,dy\,dx$$

$$= 4\int_{-3}^{3} [y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\,dx$$

$$= 4\int_{-3}^{3} 2\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx$$

$$= 8\left[\frac{x}{3}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$$

$$= 8\left[\left(0+\frac{9}{2}\cdot\frac{\pi}{2}\right) - \left(0-\frac{9}{2}\cdot\frac{\pi}{2}\right)\right]$$

$$= 8\left[\frac{9\pi}{4}+\frac{9\pi}{4}\right]$$

$$= 8\left[\frac{18\pi}{4}\right]$$

$$= 36\pi \rightarrow (1)$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = 36\pi$$

L.H.S :-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot(-\vec{k})\,dx\,dy$$

$$= \iint_{S_1} (y\vec{i}+x\vec{j}+z^2\vec{k})\,(-\vec{k})\,dx\,dy$$

$$= \iint_{S_1} -z^2\,dx\,dy \qquad (\because Z=0 \text{ on } S_1)$$

$$= 0$$

$$\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} \vec{F}\cdot\vec{k}\,dx\,dy$$

$$= \iint_{S_2} (y\vec{i}+x\vec{j}+z^2\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \iint_{S_2} z^2\,dx\,dy$$

$= \iint_{S_2} 2^2 \, dx \, dy$ $\quad (\because z = 2 \text{ on } S_2)$

$= 4(9\pi)$ $\quad (\because S_2 = \text{Area of } S_2 = \pi r^2 = \pi(3)^2 = 9\pi)$

$= 36\pi$

$$\boxed{\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = 36\pi}$$

To find $\iint_{S_3} \vec{F}\cdot\hat{n}\, ds$

Given: $x^2+y^2 = 9$

$\phi = x^2+y^2-9$

$\nabla\phi = 2x\vec{i} + 2y\vec{j} = 2(x\vec{i}+y\vec{j})$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2\sqrt{9} = 2\times 3 = 6$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i}+y\vec{j})}{6} = \frac{x\vec{i}+y\vec{j}}{3}$

$\vec{F}\cdot\hat{n} = (y\vec{i}+x\vec{j}+z^2\vec{k})\cdot\frac{1}{3}(x\vec{i}+y\vec{j})$

$= \frac{1}{3}xy + \frac{1}{3}xy$

$= \frac{2}{3}xy$

$$\boxed{\vec{F}\cdot\hat{n} = \frac{2}{3}xy}$$

Projecting the surface S_3 on the yz plane then

$ds = \frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

$$\hat{n}\cdot\vec{i} = \frac{1}{3}(x\vec{i}+y\vec{j})\cdot\vec{i}$$

$$= \frac{1}{3}x$$

$$ds = \frac{dy\,dz}{\frac{1}{3}x} = \frac{3}{x}\,dy\,dz$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_{-3}^{3} \frac{2}{3}xy\,\frac{3}{x}\,dy\,dz$$

$$(\because x^2+y^2=9$$
$$x=0 \Rightarrow y^2=9$$
$$y=\pm 3)$$

$$= \int_0^2\int_{-3}^{3} 2y\,dy\,dz = 0$$

$$\boxed{\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = 0}$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$= 0 + 36\pi + 0$$

$$= 36\pi \longrightarrow (2)$$

$$\boxed{\iint_S = 36\pi}$$

From (1) & (2), L.H.S = R.H.S

Hence, Gauss divergence theorem is verified.