



Line Integrals:

Suppose c is an arc and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the Position Vector of any point $P(x, y, z)$ on it and $\vec{f}$ is a vector Point function at P . Then $\int_c \vec{f} \cdot d\vec{r}$ is called a line integral of $\vec{f}$ over c .

Line integral $\int_A^B \vec{F} \cdot d\vec{r}$ is also known as the total work done by the force $\vec{F}$ during a displacement from A to B .

Problems:

1) Evaluate $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 y^2 \vec{i} + y \vec{j}$ and the curve c is $y^2 = 4x$ in the xy -plane from $(0, 0)$ to $(4, 4)$.

Sol: $\vec{r} = x\vec{i} + y\vec{j}$
 $d\vec{r} = dx\vec{i} + dy\vec{j}$

Given: $\vec{F} = x^2 y^2 \vec{i} + y \vec{j}$

$$\vec{F} \cdot d\vec{r} = (x^2 y^2 \vec{i} + y \vec{j}) (dx\vec{i} + dy\vec{j})$$
$$= x^2 y^2 dx + y dy$$

$$\boxed{\vec{F} \cdot d\vec{r} = x^2 y^2 dx + y dy}$$

Given $y^2 = 4x$

$$2y dy = 4 dx$$

$$y dy = 2 dx$$

$$\therefore \vec{F} \cdot d\vec{r} = x^2 y^2 dx + 2 dx = x^2 (4x) dx + 2 dx$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2) dx$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2) dx$$

$$= \left[\frac{4x^4}{4} + 2x \right]_0^4$$

$$= 4^4 + 8$$

$$= 256 + 8$$

$$= 264$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 264}$$

2) If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the straight line $y=x$ from $(0,0)$ to $(1,1)$.

Sol: $\frac{2}{3}$

3) If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the part of the curve $y = x^3$ between $x=1$ and $x=2$.

Sol: $\vec{F} = 5xy\vec{i} + 2y\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F} \cdot d\vec{r} = 5xydx + 2ydy$$

Given: $y = x^3 \Rightarrow dy = 3x^2dx$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 5x(x^3)dx + 2(x^3)3x^2dx$$

$$= \int_1^2 (5x^4 + 6x^5)dx$$



$$= \left[\frac{5x^5}{5} + \frac{6x^6}{6} \right]_1^2$$

$$= [32 + 64 - (1 + 1)]$$

$$= 94$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 94}$$

4) Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (xyz-x)\vec{k}$ along the line joining the points $(0,0,0)$ to $(2,1,1)$.

Sol: $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (xyz-x)\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (2y+3)dx + xzdy + (xyz-x)dz$$

The equation of line joining the points $(0,0,0)$ and $(2,1,1)$ is

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$$x = 2t, y = t, z = t$$

$$\vec{F} \cdot d\vec{r} = 2(2t+3)dt + 2t^2dt + (t^3-2t)dt$$

$$\boxed{\vec{F} \cdot d\vec{r} = (3t^2 + 2t + 6)dt}$$

$$\text{At } x=0, y=0, z=0 \Rightarrow t=0$$

$$\text{At } x=2, y=1, z=1 \Rightarrow t=1$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 2t + 6)dt = [t^3 + t^2 + 6t]_0^1 = 8$$



5) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x=t, y=t^2, z=t^3$.

Sol: $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$
 $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$

Given: $x=t, y=t^2, z=t^3$

$$dx=dt, dy=2tdt, dz=3t^2dt$$

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= (3t^2 + 6t^2)dt - 14(t^2 \cdot t^3)2tdt + 20(t \cdot t^6)3t^2dt \\ &= (9t^2 - 28t^6 + 60t^9)dt\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 5$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 5}$$

Surface Integral:

Let S be a Surface whose Projection R_{xy} on the xy -plane is such that the points on S have a 1-1 correspondance with the points on R_{xy} . Let ds be a vector element of the area.

Then

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} ds = \iint_{R_{xy}} \vec{F} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \vec{k}|}$$



For yz -plane,
$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{R_{yz}} \vec{F} \cdot \hat{n} \frac{dy \, dz}{|\hat{n} \cdot \vec{j}|}$$

For xz -plane,
$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{R_{xz}} \vec{F} \cdot \hat{n} \frac{dx \, dz}{|\hat{n} \cdot \vec{j}|}$$

The Surface integral $\iint_S \vec{F} \cdot d\vec{s}$ represents the total flux of $\vec{F}$ through the whole surface.