



## VECTOR CALCULUS

Scalar quantity :

A scalar quantity is that which has magnitude and is not related to any direction.

Vector quantity :

A vector quantity is that which has both magnitude and direction.

Scalar point function :

If corresponding to each point  $P$  of a region  $R$  there corresponds a scalar denoted by  $\phi(P)$  or  $\phi(x, y, z)$  then  $\phi$  is said to be a scalar point function for the region  $R$ .

Example :

The temperature  $\phi(P)$  at any point  $P$  of a body occupying a certain region is a scalar point function.

Vector point function :

If corresponding to each point  $P$  of a region  $R$ , there corresponds a vector denoted by  $F(P)$  then  $F$  is said to be a vector point function for the region  $R$ .

Example : The acceleration  $F(P)$  of a particle at any time  $t$  occupying the position  $P$  in a certain region is a vector point function.

Vector Differential operator:

The vector differential operator  $\nabla$  is defined as,

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} = \sum \vec{i} \frac{\partial}{\partial x}$$

Gradient of a scalar point function:

Let  $\phi(x, y, z)$  be a scalar point function and is continuously differentiable then the vector

$$\nabla \phi = \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \phi$$

$$= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

is called the gradient of  $\phi$  and is written as

grad  $\phi$       i.e., grad  $\phi = \nabla \phi$

Note:

1.  $\nabla \phi$  defines a vector field
2.  $\nabla \phi = \phi \nabla$ . There will be no '.' or 'x' between  $\phi$  and  $\nabla$ .

Properties of Gradient:-

1. If  $f$  and  $g$  are two scalar point functions

then,

$$\nabla (f \pm g) = \nabla f \pm \nabla g$$

$$\text{or grad } (f \pm g) = \text{grad } f \pm \text{grad } g$$

2. If  $f$  and  $g$  are two scalar point functions then

$$\therefore \nabla(fg) = f \nabla g + g \nabla f$$

$$\text{(or) grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$$

3. If  $f$  and  $g$  are two scalar point functions then,

$$\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2} \quad \text{where } g \neq 0$$

$$\text{(or) grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}$$

4. Gradient of a constant is zero.

$$\nabla \phi = 0.$$

Problems:

1. Find grad  $\phi$  where  $\phi = x^2 + y^2 + z^2$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \vec{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) + \vec{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$= \vec{i} (2x) + \vec{j} (2y) + \vec{k} (2z)$$

$$\nabla \phi = 2x \vec{i} + 2y \vec{j} + 2z \vec{k}$$

a. Find grad  $\phi$  if  $\phi = xyz$  at  $(1, 1, 1)$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \vec{i} \frac{\partial}{\partial x} (xyz) + \vec{j} \frac{\partial}{\partial y} (xyz) + \vec{k} \frac{\partial}{\partial z} (xyz)$$

$$= \vec{i} (yz) + \vec{j} (xz) + \vec{k} (xy)$$

$$\nabla \varphi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$

3. Find grad  $\varphi$  where  $\varphi = 3x^2y - y^3z^2$  at  $(1,1,1)$

$$\nabla \varphi = \vec{i} \frac{\partial \varphi}{\partial x} + \vec{j} \frac{\partial \varphi}{\partial y} + \vec{k} \frac{\partial \varphi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \vec{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2)$$

$$+ \vec{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2)$$

$$= \vec{i} (6xy) + \vec{j} (3x^2 - 3y^2z^2) + \vec{k} (-2y^3z)$$

$$\nabla \varphi_{(1,1,1)} = 6\vec{i} + \vec{k} (-2)$$

$$= 6\vec{i} - 2\vec{k}$$

4. If  $\varphi = \log(x^2+y^2+z^2)$  find  $\nabla \varphi$

$$\nabla \varphi = \vec{i} \frac{\partial \varphi}{\partial x} + \vec{j} \frac{\partial \varphi}{\partial y} + \vec{k} \frac{\partial \varphi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} [\log(x^2+y^2+z^2)] + \vec{j} \frac{\partial}{\partial y} (\log(x^2+y^2+z^2))$$

$$+ \vec{k} \frac{\partial}{\partial z} (\log(x^2+y^2+z^2))$$

$$= \vec{i} \frac{1}{x^2+y^2+z^2} (2x) + \vec{j} \frac{1}{x^2+y^2+z^2} (2y) + \vec{k} \frac{1}{x^2+y^2+z^2} (2z)$$

$$= \frac{2}{x^2+y^2+z^2} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla \varphi = \frac{2\vec{r}}{x^2+y^2+z^2}$$

5) Find  $\nabla(\log r)$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\nabla(\log r) = \vec{i} \frac{\partial}{\partial x}(\log r) + \vec{j} \frac{\partial}{\partial y}(\log r) + \vec{k} \frac{\partial}{\partial z}(\log r)$$

$$= \vec{i} \frac{1}{r} \frac{\partial r}{\partial x} + \vec{j} \frac{1}{r} \frac{\partial r}{\partial y} + \vec{k} \frac{1}{r} \frac{\partial r}{\partial z} \rightarrow \text{①}$$

since  $r = \sqrt{x^2 + y^2 + z^2}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

Similarly,  $\frac{\partial r}{\partial y} = \frac{y}{r}$  &  $\frac{\partial r}{\partial z} = \frac{z}{r}$

sub the values in ①

$$\nabla(\log r) = \vec{i} \frac{1}{r} \left(\frac{x}{r}\right) + \vec{j} \frac{1}{r} \left(\frac{y}{r}\right) + \vec{k} \frac{1}{r} \left(\frac{z}{r}\right)$$

$$= \frac{1}{r^2} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla(\log r) = \frac{\vec{r}}{r^2}$$

⑥ If  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  such that  $|\vec{r}| = r$ ,

Prove that

i)  $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

ii)  $\nabla \left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$

iii)  $\nabla r^n = n r^{n-2} \vec{r}$

iv)  $\nabla f(r) = f'(r) \nabla r$

v)  $\nabla f(r) \times \vec{r} = 0$

vi) If  $\nabla \phi$  is solenoidal find  $\nabla^2 \phi$

Solution:

i) Given:  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \vec{i} \frac{\partial r}{\partial x} + \vec{j} \frac{\partial r}{\partial y} + \vec{k} \frac{\partial r}{\partial z}$$

$$= \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r}$$

$$= \frac{1}{r} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla r = \frac{\vec{r}}{r} = \hat{r}}$$

ii)  $\nabla \left(\frac{1}{r}\right) = \vec{i} \frac{\partial}{\partial x} \left(\frac{1}{r}\right) + \vec{j} \frac{\partial}{\partial y} \left(\frac{1}{r}\right) + \vec{k} \frac{\partial}{\partial z} \left(\frac{1}{r}\right)$

$$= \vec{i} \left(\frac{-1}{r^2} \frac{\partial r}{\partial x}\right) + \vec{j} \left(\frac{-1}{r^2} \frac{\partial r}{\partial y}\right) + \vec{k} \left(\frac{-1}{r^2} \frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2} \left[ \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$$

$$\nabla \left(\frac{1}{r}\right) = \frac{-1}{r^3} (\vec{r}) = \frac{-1}{r^2} \left(\frac{\vec{r}}{r}\right) = \frac{-\hat{r}}{r^2}$$

$$\nabla \left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

$$\begin{aligned}
 \text{iii) } \nabla r^n &= \left( \vec{i} \frac{\partial r^n}{\partial x} + \vec{j} \frac{\partial r^n}{\partial y} + \vec{k} \frac{\partial r^n}{\partial z} \right) \\
 &= \vec{i} n r^{n-1} \frac{\partial r}{\partial x} + \vec{j} n r^{n-1} \frac{\partial r}{\partial y} + \vec{k} n r^{n-1} \frac{\partial r}{\partial z} \\
 &= n r^{n-1} \left[ \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right] \\
 &= \frac{n r^{n-1}}{r} [x\vec{i} + y\vec{j} + z\vec{k}]
 \end{aligned}$$

$$\nabla r^n = n r^{n-2} \vec{r}$$

$$\begin{aligned}
 \text{iv) } \nabla f(r) &= \vec{i} \frac{\partial f(r)}{\partial x} + \vec{j} \frac{\partial f(r)}{\partial y} + \vec{k} \frac{\partial f(r)}{\partial z} \\
 &= \vec{i} f'(r) \frac{\partial r}{\partial x} + \vec{j} f'(r) \frac{\partial r}{\partial y} + \vec{k} f'(r) \frac{\partial r}{\partial z} \\
 &= f'(r) \left[ \vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right] \\
 &= \frac{f'(r)}{r} (x\vec{i} + y\vec{j} + z\vec{k})
 \end{aligned}$$

$$\nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

$$\begin{aligned}
 \text{v) } \nabla f(r) \times \vec{r} &= \frac{f'(r)}{r} \vec{r} \times \vec{r} \\
 &= \frac{1}{r} f'(r) [\vec{r} \times \vec{r}] = 0. \quad \because \vec{r} \times \vec{r} = 0
 \end{aligned}$$

$$\therefore \nabla f(r) \times \vec{r} = 0.$$

$$\begin{aligned}
 \text{vi) } \nabla^2 \phi &= \nabla(\nabla \phi) & (\because \nabla \phi \text{ is solenoidal } \nabla \phi = 0) \\
 &= \nabla(0) \\
 &\nabla^2 \phi = 0.
 \end{aligned}$$

Level Surface : Important Results

Unit Normal

A unit normal to the given surface  $\phi$  at the point is  $\frac{\nabla\phi}{|\nabla\phi|}$

Directional Derivative:-

The directional derivative of  $\phi$  in the direction  $\vec{a}$  is given by

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} \text{ (or) } \nabla\phi \cdot \hat{n} \text{ where } \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface.

Its maximum value is  $|\nabla\phi|$

Angle between two surfaces:

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

Note!

If the surfaces cut orthogonally then,

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0.$$



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Problems:

1. Find a unit normal to the surface  $x^2y + 2xz = 4$  at  $(2, -2, 3)$

soln:  $\phi = x^2y + 2xz - 4$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2y + 2xz - 4) + \vec{j} \frac{\partial}{\partial y} (x^2y + 2xz - 4) + \vec{k} \frac{\partial}{\partial z} (x^2y + 2xz - 4)$$

$$= \vec{i} (2xy + 2z) + \vec{j} (x^2) + \vec{k} (2x)$$

$$\nabla\phi_{(2, -2, 3)} = \vec{i} (-8 + 6) + \vec{j} (4) + \vec{k} (4)$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

Unit normal to the given surface at  $(2, -2, 3)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3} (-\vec{i} + 2\vec{j} + 2\vec{k})$$

2. Find the unit normal to  $x^2 - y^2 + z = 2$  at  $(1, -1, 2)$

Solution:

$$\phi = x^2 - y^2 + z - 2$$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 - y^2 + z - 2) + \vec{j} \frac{\partial}{\partial y} (x^2 - y^2 + z - 2) + \vec{k} \frac{\partial}{\partial z} (x^2 - y^2 + z - 2)$$

$$= \vec{i} (2x) + \vec{j} (-2y) + \vec{k} (1)$$

$$= 2x\vec{i} + \vec{k} - 2y\vec{j}$$

$$\begin{aligned}\nabla\phi_{(1,-1,2)} &= \vec{i}(2(1)) - \vec{j}(2(-1)) + \vec{k}(1) \\ &= 2\vec{i} + 2\vec{j} + \vec{k}\end{aligned}$$

$$\begin{aligned}|\nabla\phi| &= \sqrt{2^2 + (2)^2 + 1^2} = \sqrt{4+4+1} \\ &= \sqrt{9} \\ &= 3.\end{aligned}$$

$$\bullet \quad \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

3. Find the unit vector normal to  $x^2 + xy + z^2 = 4$   
at  $(1, -1, 2)$

$$\begin{aligned}\nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \\ &= i(2x+y) + \vec{j}(x) + \vec{k}(2z)\end{aligned}$$

$$\begin{aligned}\nabla\phi_{(1,-1,2)} &= \vec{i}(2(1)+(-1)) + \vec{j}(1) + \vec{k}(2(2)) \\ &= \vec{i} + \vec{j} + 4\vec{k}\end{aligned}$$

$$|\nabla\phi| = \sqrt{1+1+16} = \sqrt{18}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$

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4. Find the directional derivative of the function  $x^2 + 2xy$  at  $(1, -1, 3)$  in the direction  $\vec{i} + 2\vec{j} + 2\vec{k}$

Soln:  $\phi = x^2 + 2xy$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 + 2xy) + \vec{j} \frac{\partial}{\partial y} (x^2 + 2xy) + \vec{k} \frac{\partial}{\partial z} (x^2 + 2xy)$$

$$= \vec{i} (2x + 2y) + \vec{j} (2x) + \vec{k} (0)$$

$$\nabla\phi_{(1, -1, 3)} = \vec{i} (2(1) + 2(-1)) + \vec{j} (2(1))$$

$$= 2\vec{j}$$

Given:  $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$\nabla\phi \cdot \hat{n} = 2\vec{j} \cdot \left[ \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} \right] = \frac{4}{3}$$

$$\therefore \nabla\phi \cdot \hat{n} = \frac{4}{3}$$

5. Find the directional derivative of  $xy + yz + zx$  at  $(1, 1, 1)$  in the direction  $\vec{i} + \vec{j} + 2\vec{k}$

$$\phi = xy + yz + zx$$

$$\nabla\phi = \vec{i} \frac{\partial}{\partial x} (xy + yz + zx) + \vec{j} \frac{\partial}{\partial y} (xy + yz + zx) + \vec{k} \frac{\partial}{\partial z} (xy + yz + zx)$$

$$= \vec{i} (y + z) + \vec{j} (x + z) + \vec{k} (y + x)$$

$$\nabla\phi_{(1, 1, 1)} = 2\vec{i} + 2\vec{j} + 2\vec{k} - 2(\vec{i} + \vec{j} + \vec{k})$$

Given:  $\vec{a} = \vec{i} + \vec{j}$

$$|\vec{a}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + \vec{j}}{\sqrt{2}}$$

$$\begin{aligned} \text{Directional derivative} &= \nabla \phi \cdot \hat{n} \\ &= 2(\vec{i} + \vec{j} + \vec{k}) \cdot \frac{\vec{i} + \vec{j}}{\sqrt{2}} \\ &= \frac{2(1+1)}{\sqrt{2}} \\ &= 2\sqrt{2} \end{aligned}$$

⑥ Find the directional derivative of  $3x^2 + 2y - 3z$  at  $(1, 1, 1)$  in the direction  $2\vec{i} + 2\vec{j} - \vec{k}$

$$\begin{aligned} \phi &= 3x^2 + 2y - 3z \\ \nabla \phi &= \vec{i} \frac{\partial}{\partial x}(3x^2 + 2y - 3z) + \vec{j} \frac{\partial}{\partial y}(3x^2 + 2y - 3z) + \vec{k} \frac{\partial}{\partial z}(3x^2 + 2y - 3z) \end{aligned}$$

$$= \vec{i}(6x) + \vec{j}(2) + \vec{k}(-3)$$

$$= 6x\vec{i} + 2\vec{j} - 3\vec{k}$$

$$\nabla \phi_{(1,1,1)} = 6\vec{i} + 2\vec{j} - 3\vec{k}$$

Given:  $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k}$      $|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

$$\nabla \phi \cdot \hat{n} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3} \cdot 6\vec{i} + 2\vec{j} - 3\vec{k}$$

$$= \frac{12 + 4 + 3}{3} = \frac{19}{3}$$

7. What is the greatest rate of increase of  $\phi = xyz^2$  at  $(1, 0, 3)$ ?

Let  $\phi = xyz^2$

$$\nabla\phi = \vec{i} \frac{\partial}{\partial x}(xyz^2) + \vec{j} \frac{\partial}{\partial y}(xyz^2) + \vec{k} \frac{\partial}{\partial z}(xyz^2)$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$\nabla\phi_{(1,0,3)} = 9\vec{j}$$

Maximum (or) Greatest rate of increase =  $|\nabla\phi| = \sqrt{9^2} = 9$

8. In what direction from the point  $(1, -1, 2)$  is the directional derivative of  $\phi = x^2y^2z^2$  a maximum? What is the magnitude of this maximum?  $x^2y^2z^2$  at  $(1, 2, -1)$

$\phi = x^2y^2z^2$

$$\nabla\phi = \vec{i} \frac{\partial}{\partial x}(x^2y^2z^2) + \vec{j} \frac{\partial}{\partial y}(x^2y^2z^2) + \vec{k} \frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= 2xy^2z^2\vec{i} + 2yx^2z^2\vec{j} + 2x^2y^2z\vec{k}$$

$$\nabla\phi_{(1,-1,2)} = 2(1)(-1)^2(2)^2\vec{i} + 2(-1)(1)^2(2)^2\vec{j} + 2(1)^2(-1)^2(2)\vec{k}$$

$$= 8\vec{i} - 8\vec{j} + 4\vec{k} \text{ is the directional derivative}$$

Magnitude is  $|\nabla\phi| = \sqrt{8^2 + (-8)^2 + 4^2}$

$$= \sqrt{64 + 64 + 16}$$

$$= 12.$$

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Q Find the directional derivative of  $\phi = xy^2z^3$  at the point  $(1, 1, 1)$  along the normal to the surface  $x^2 + xy + z^2 = 3$  at the point  $(1, 1, 1)$

$\nabla\phi$  is normal to the surface  $x^2 + xy + z^2 = 3$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x}(x^2 + xy + z^2 - 3) + \vec{j} \frac{\partial}{\partial y}(x^2 + xy + z^2 - 3) + \vec{k} \frac{\partial}{\partial z}(x^2 + xy + z^2 - 3)$$

$$= \vec{i}(2x + y) + \vec{j}(x) + \vec{k}(2z)$$

$$\nabla\phi_{(1,1,1)} = 3\vec{i} + \vec{j} + 2\vec{k}$$

To find the directional derivative of  $\phi = xy^2z^3$  at  $(1, 1, 1)$  in the direction  $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$

$$\nabla\phi = \vec{i} \frac{\partial}{\partial x}(xy^2z^3) + \vec{j} \frac{\partial}{\partial y}(xy^2z^3) + \vec{k} \frac{\partial}{\partial z}(xy^2z^3)$$

$$= \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$$

$$\nabla\phi_{(1,1,1)} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{Directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot \frac{(3\vec{i} + \vec{j} + 2\vec{k})}{\sqrt{3^2 + 1^2 + 2^2}}$$

$$= \frac{3 + 2 + 6}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

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10. Find the angle between the surfaces  $x^2 + y^2 + z^2 = 5$  and  $x^2 + y^2 + z^2 - 2x = 5$  at  $(0, 1, 2)$

Let  $\phi_1 = x^2 + y^2 + z^2 - 5$  ;  $\phi_2 = x^2 + y^2 + z^2 - 2x - 5$

$$\frac{\partial \phi_1}{\partial x} = 2x$$

$$\frac{\partial \phi_2}{\partial x} = 2x - 2$$

$$\frac{\partial \phi_1}{\partial y} = 2y$$

$$\frac{\partial \phi_2}{\partial y} = 2y$$

$$\frac{\partial \phi_1}{\partial z} = 2z$$

$$\frac{\partial \phi_2}{\partial z} = 2z$$

$$\nabla \phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \quad ; \quad \nabla \phi_2 = (2x-2)\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla \phi_1(0, 1, 2) = 2\vec{j} + 4\vec{k} \quad \nabla \phi_2(0, 1, 2) = -2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$|\nabla \phi_1| = \sqrt{4+16} = \sqrt{20}$$

$$|\nabla \phi_2| = \sqrt{4+4+16} = \sqrt{24}$$

Angle between the surfaces,

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$= \frac{(2\vec{j} + 4\vec{k}) \cdot (-2\vec{i} + 2\vec{j} + 4\vec{k})}{\sqrt{20} \cdot \sqrt{24}}$$

$$= \frac{4+16}{\sqrt{20} \cdot \sqrt{24}} = \frac{20}{\sqrt{20} \cdot \sqrt{24}} = \frac{\sqrt{20}}{\sqrt{24}}$$

$$\cos \theta = \sqrt{\frac{5}{6}}$$

$$\theta = \cos^{-1} \sqrt{\frac{5}{6}}$$

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11. Find the angle between the surfaces  $x \log z = y^2 - 1$   
and  $x^2 y = 2 - z$  at the point  $(1, 1, 1)$

$$\varphi_1 = x \log z - y^2 + 1 \quad ; \quad \varphi_2 = x^2 y - 2 + z \quad \text{at } (1, 1, 1)$$

$$\frac{\partial \varphi_1}{\partial x} = \log z$$

$$\frac{\partial \varphi_2}{\partial x} = 2xy$$

$$\frac{\partial \varphi_1}{\partial y} = -2y$$

$$\frac{\partial \varphi_2}{\partial y} = x^2$$

$$\frac{\partial \varphi_1}{\partial z} = \frac{x}{z}$$

$$\frac{\partial \varphi_2}{\partial z} = 1$$

$$\nabla \varphi_1 = \log z \vec{i} - 2y \vec{j} + \frac{x}{z} \vec{k} \quad ; \quad \nabla \varphi_2 = 2xy \vec{i} + x^2 \vec{j} + \vec{k}$$

$$\nabla \varphi_1(1,1,1) = -2\vec{j} + \vec{k}$$

$$\nabla \varphi_2(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla \varphi_1| = \sqrt{4+1} = \sqrt{5}$$

$$|\nabla \varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

$$\cos \theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla \varphi_1| |\nabla \varphi_2|}$$

$$= \frac{(-2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{5}\sqrt{6}}$$

$$= \frac{-1}{\sqrt{5}\sqrt{6}}$$

$$\cos \theta = \frac{-1}{\sqrt{30}}$$

$$\theta = \cos^{-1} \left( \frac{-1}{\sqrt{30}} \right)$$



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18. Find the angle between the surfaces  $x^2 + y^2 + z^2 = 9$

$z = x^2 + y^2 - 2$  at  $(2, -1, 2)$

$\phi_1 = x^2 + y^2 + z^2 - 9$  ;  $\phi_2 = x^2 + y^2 - 2 - z$

$\nabla \phi_1 \Rightarrow$

$\frac{\partial \phi_1}{\partial x} = 2x$

$\frac{\partial \phi_2}{\partial x} = 2x$

$\frac{\partial \phi_1}{\partial y} = 2y$

$\frac{\partial \phi_2}{\partial y} = 2y$

$\frac{\partial \phi_1}{\partial z} = 2z$

$\frac{\partial \phi_2}{\partial z} = -1$

$\nabla \phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\nabla \phi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$

$\nabla \phi_1(2, -1, 2) = 4\vec{i} - 2\vec{j} + 4\vec{k}$

$\nabla \phi_2(2, -1, 2) = 4\vec{i} - 2\vec{j} - \vec{k}$

$|\nabla \phi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$  ;  $|\nabla \phi_2| = \sqrt{16+4+1} = \sqrt{21}$

~~cos~~  $\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$

$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{(6)(\sqrt{21})}$

$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$

~~cos~~  $\cos \theta = \frac{8}{3\sqrt{21}}$

$\theta = \cos^{-1} \left( \frac{8}{3\sqrt{21}} \right)$

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13. Find  $a$  and  $b$  such that surfaces  $ax^2 + byz = (a+2)z$  and  $4x^2y + z^3 = 4$  cuts orthogonally at  $(1, -1, 2)$ .

$$\text{Let } \varphi_1 = ax^2 + byz - (a+2)z$$

$$\varphi_2 = 4x^2y + z^3 - 4$$

$$\begin{aligned} \nabla\varphi_1 &= \vec{i} \frac{\partial}{\partial x}(ax^2 + byz - (a+2)z) + \vec{j} \frac{\partial}{\partial y}(ax^2 + byz - (a+2)z) \\ &\quad + \vec{k} \frac{\partial}{\partial z}(ax^2 + byz - (a+2)z) \\ &= \vec{i}(2ax - a - 2) + \vec{j}(bz) + \vec{k}(by) \end{aligned}$$

$$\nabla\varphi_1(1, -1, 2) = (a-2)\vec{i} + 2b\vec{j} - b\vec{k}$$

$$\begin{aligned} \nabla\varphi_2 &= \vec{i} \frac{\partial}{\partial x}(4x^2y + z^3 - 4) + \vec{j} \frac{\partial}{\partial y}(4x^2y + z^3 - 4) + \vec{k} \frac{\partial}{\partial z}(4x^2y + z^3 - 4) \\ &= \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2) \end{aligned}$$

$$\nabla\varphi_2(1, -1, 2) = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Since the surface cuts orthogonally,

$$\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$$

$$[(a-2)\vec{i} + 2b\vec{j} - b\vec{k}] \cdot [-8\vec{i} + 4\vec{j} + 12\vec{k}] = 0$$

$$-8a + 16 + 8b + 12b = 0$$

$$-8a - 8b - 8a + 4b = -16$$

$$2a + b = 4 \rightarrow \textcircled{1}$$

Since the point  $(1, -1, 2)$  lies on  $\varphi_1$

$$a - 2b - (a+2) = 0 \rightarrow \boxed{b = -1}$$

$$\text{Sub } b = -1 \text{ in } \textcircled{1} \rightarrow \boxed{a = 5}$$

14. Find the values of  $a$  and  $b$  so that the surface  
 $ax^3 - by^2z = (a+3)x^2$  and  $4x^2y - z^3 = 11$  may cut orthogonally  
 at  $(2, -1, -3)$

Let  $\phi_1 = ax^3 - by^2z - (a+3)x^2$ ;  $\phi_2 = 4x^2y - z^3 - 11$

$$\nabla\phi_1 = \vec{i}(3ax^2 - 2(a+3)x) + \vec{j}(-6b) + \vec{k}(-b)$$

$$\begin{aligned} \nabla\phi_1(2, -1, -3) &= \vec{i}(12a - 4a - 12) + \vec{j}(-6b) + \vec{k}(-b) \\ &= \vec{i}(8a - 12) + \vec{j}(-6b) + \vec{k}(-b) \end{aligned}$$

$$\nabla\phi_2 = \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(-3z^2)$$

$$\nabla\phi_2(2, -1, -3) = -16\vec{i} + 16\vec{j} - 27\vec{k}$$

$\therefore$  The surface cuts orthogonally

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$[(8a-12)\vec{i} - 6b\vec{j} - b\vec{k}] \cdot [-16\vec{i} + 16\vec{j} - 27\vec{k}] = 0$$

$$(8a-12)(-16) - 6b(16) + 27b = 0$$

$$-128a + 192 - 96b + 27b = 0.$$

$$-128a - 69b + 192 = 0$$

$$128a + 69b = 192 \rightarrow \textcircled{1}$$

$\therefore$  The point  $(2, -1, -3)$  lie on  $\phi_1$ ,

$$8a + 3b - 4(a+3) = 0$$

$$8a + 3b - 4a - 12 = 0$$

$$\Rightarrow 4a + 3b = 12 \rightarrow \textcircled{2}$$

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⊗ by 23 in ⊙

$$\rightarrow 128a + 69/b = 192$$

$$\begin{array}{r} (-) 92a + 69/b = 276 \\ \hline \end{array}$$

$$36a = -84$$

$$a = \frac{-84}{36} = \frac{-7}{3}$$

$$\therefore a = \frac{-7}{3}$$

Sub a in ⊙  $\rightarrow 4\left(\frac{-7}{3}\right) + 3b = 12$

$$-\frac{28}{3} + 3b = 12$$

$$3b = 12 + \frac{28}{3}$$

$$= \frac{36 + 28}{3}$$

$$b = \frac{64}{9}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

find the angle b/w the normals to the surface  $xy = z^2$  at the pt  $(1, 4, 2)$ ,  $(-3, -3, 3)$

$$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\nabla\phi_{(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{33}$$

$$\nabla\phi_2 = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$|\nabla\phi_2| = \sqrt{54}$$

$$\cos\theta = \frac{9}{\sqrt{33}\sqrt{54}} = \frac{1}{\sqrt{22}}$$

$$= \frac{-12 - 32 + 12}{\sqrt{33}\sqrt{54}} = \frac{9}{\sqrt{33 \times 2 \times 3 \times 3 \times 3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$