Surface Integral:-

Let S be a surface whose projection R_{xy} on the xyplane is ∋ the points on s have a one-one correspondence with the points on R_{xy}. Let ds be a vector element of the area. then

$$\iint_S \vec{F}\cdot\vec{ds} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xy}} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} \Rightarrow \text{for xy plane}$$

$$\iint_S \vec{F}\cdot\vec{ds} = \iint_S \vec{F}\cdot\hat{n}\,ds \Rightarrow \iint_{R_{yz}} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|} \quad \text{for yz plane}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xz}} \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\vec{j}|} \quad \text{for xz plane}$$

Line Integrals:

Suppose c is an arc and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of any point $P(x, y, z)$ on it and $\vec{F}$ is a vector point function at P. Then

$\int_C \vec{F}\cdot d\vec{r}$ is called a line integral of $\vec{F}$ over c.

Line integral $\int_A^B \vec{F}\cdot d\vec{r}$ is also known as the total work done by the force $\vec{F}$ during a displacement from A to B.

① Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ in the xy-plane from (0,0) to (4,4)

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

Given: $\vec{F} = x^2y^2\vec{i} + y\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2y^2\vec{i} + y\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$

$= x^2y^2\,dx + y\,dy$

Given: $y^2 = 4x$

$2y\,dy = 4\,dx$

$y\,dy = 2\,dx$

$\therefore \vec{F}\cdot d\vec{r} = x^2y^2\,dx + 2\,dx$

$= x^2(4x)\,dx + 2\,dx$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^4 (4x^3 + 2)\,dx$

$= \left[\frac{4x^4}{4} + 2x\right]_0^4 = 4^4 + 2(4)$

$= 256 + 8 = 264$

② If $\vec{F} = x^2\vec{i} + xy\,\vec{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the straight line $y = x$ from (0,0) to (1,1)

$\vec{r} = x\vec{i} + y\vec{j} \rightarrow d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

Given:- $\vec{F} = x^2\vec{i} + xy\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2\vec{i} + xy\,\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$

$= x^2\,dx + xy\,dy$

Also given: $y = x \Rightarrow dy = dx$.

$$\therefore \vec{F}\cdot d\vec{r} = x^2dx + x(x)(dx)$$
$$= x^2dx + x^2dx$$
$$= 2x^2dx$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 2x^2dx$$
$$= 2\int_0^1 x^2dx = 2\left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{2}{3}.$$

③ If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$. evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the part of the curve $y = x^3$. between $x=1$ and $x=2$.

$$\vec{F} = 5xy\vec{i} + 2y\vec{j}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j}$$
$$\vec{F}\cdot d\vec{r} = 5xydx + 2ydy$$

Given :- $y = x^3 \Rightarrow dy = 3x^2dx$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^2 (5x(x^3)dx + 2(x^3)3x^2dx)$$
$$= \int_1^2 (5x^4 + 6x^5)dx = \left(\frac{5x^5}{5} + \frac{6x^6}{6}\right)_1^2$$
$$= [(2)^5 + (2)^6] - [1^5 + 1^6]$$
$$= 32 + 64 - 2$$
$$= 94.$$

④ Find $\int_C \vec{F}d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$ along the line joining the pts $(0,0,0)$ to $(2,1,1)$ Ans: 8

$$\vec{F}\cdot d\vec{r} = (2y+3)dx + xzdy + (yz - x)dz$$

The eqn of the line joining the pts $(0,0,0)$ & $(2,1,1)$ is

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = k \text{ (say)}$$

$x = 2t,\ y = t,\ z = t \Rightarrow dx = 2dt,\ dy = dt,\ dz = dt$

$$\vec{F}\cdot d\vec{r} = (2t+3)(2dt) + 2t^2 dt + (t^2 - 2t)dt$$

$$= (4t + 6 + 2t^2 + t^2 - 2t)\,dt = (3t^2 + 2t + 6)\,dt$$

For $(0,0,0) \Rightarrow t = 0$ & $(2,1,1) \Rightarrow t = 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2 + 2t + 6)\,dt = \left[t^3 + t^2 + 6t\right]_0^1$$

$$= 1 + 1 + 6 = 8$$

5. If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x = t;\ y = t^2;\ z = t^3$

$$\vec{F}\cdot d\vec{r} = (3x^2 + 6y)dx - 14yz\,dy + 20xz^2 dz$$

Given :- $x = t,\ y = t^2,\ z = t^3$ & $dx = dt,\ dy = 2t\,dt,\ dz = 3t^2 dt$

$$\vec{F}\cdot d\vec{r} = (3t^2 + 6t^2)\,dt - 14t^5 \cdot 2t\,dt + 20t\cdot t^6\, 3t^2 dt$$

$$= (9t^2 - 28t^6 + 60t^9)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= \left[3t^4 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 3 - 4 + 6$$

$$= 5.$$