



Stoke's Theorem:

If \vec{F} is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

where \hat{n} is the unit normal vector at any point of S .

Note:

1. If \vec{F} is irrotational, $\nabla \times \vec{F} = 0$.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 0 \quad \& \text{ hence } \vec{F} \text{ is conservative}$$

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C Pdx + Qdy + Rdz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

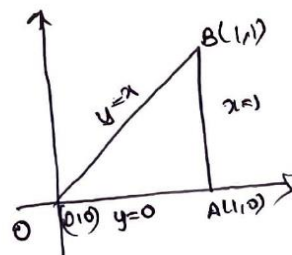
Problems:-

① Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$

and C is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$

By Stoke's theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



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Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane

$$\vec{F} = y^2 \vec{i} + x^2 y \vec{j} - (x+2) \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 y & -(x+2) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

Since the triangle lies on xy plane and hence the unit vector normal to the surface OAB is \vec{k} .

i.e) $\hat{n} = \vec{k}$

$$\nabla \times \vec{F} \cdot \hat{n} = [\vec{j} + 2(x-y)\vec{k}] \cdot \vec{k} = 2(x-y)$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = \iint_S 2(x-y) \, dx \, dy$$

(∵ S lies on xy plane
 $ds = dx \, dy$)

$$= 2 \int_0^1 \int_0^1 (x-y) \, dx \, dy$$

[x → y to 1
y → 0 to 1

$$= 2 \int_0^1 \left[\frac{x^2}{2} - xy \right]_0^1 dy$$

$$= 2 \int_0^1 \left[\frac{1}{2} - y - \frac{y^2}{2} + y^2 \right] dy$$

$$= 2 \left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3} \right]_0^1$$

$$= 2 \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3} \right) = \frac{1}{3}$$

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Q. Verify Stoke's theorem for a vector field defined by
 $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane
 bounded by the lines $x=0, x=a, y=0$ and $y=b$

By Stoke's theorem,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

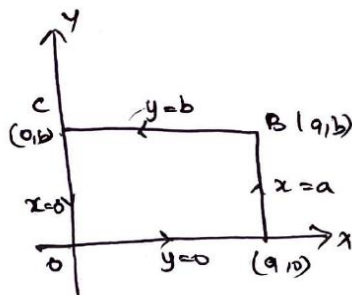
RHS: Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y) = 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is \vec{k}

(i) $\hat{n} = \vec{k}$

$$\begin{aligned} \therefore \text{curl } \vec{F} \cdot \hat{n} \, ds &= 4y\vec{k} \cdot \vec{k} \, dxdy \\ &= 4y \, dxdy. \end{aligned}$$



$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds &= \iint_S 4y \, dxdy = \int_0^b \int_0^a 4y \, dxdy \\ &= 4 \int_0^b [x]_0^a dy \\ &= 4a \left[\frac{y^2}{2} \right]_0^b = \frac{4ab^2}{2} \\ &= 2ab^2 \rightarrow \textcircled{1} \end{aligned}$$