LINE INTEGRALS :

Suppose C is an arc and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of any point $P(x, y, z)$ on it and $\vec{f}$ is a vector point function at P. Then $\int_C \vec{f} \cdot d\vec{r}$ is called a line integral of $\vec{f}$ over C.

Line integral $\int_A^B \vec{F} \cdot d\vec{r}$ is also known as the total work done by the force $\vec{F}$ during a displacement from A to B.

(1) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2 y^2 \vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ in the xy-plane from $(0,0)$ to $(4,4)$.

Soln:

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

Given : $\vec{F} = x^2 y^2 \vec{i} + y\vec{j}$

$$\vec{F} \cdot d\vec{r} = (x^2 y^2 \vec{i} + y\vec{j}) \cdot (dx\,\vec{i} + dy\,\vec{j})$$

$$= x^2 y^2 dx + y\,dy$$

Given : $y^2 = 4x$

$$2y\,dy = 4\,dx$$

$$y\,dy = 2\,dx$$

$$\therefore \vec{F} \cdot d\vec{r} = x^2 y^2 dx + 2\,dx = x^2 (4x)\,dx + 2\,dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3 + 2)\,dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^4 (4x^3+2)\,dx$$

$$= \left[4\frac{x^4}{4} + 2x\right]_0^4$$

$$= 4^4 + 8 = 256 + 8$$

$$= 264$$

(2) If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the straight line $y = x$ from $(0,0)$ to $(1,1)$.

Soln: $2/3$.

(3) If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the part of the curve $y = x^3$ between $x=1$ and $x=2$.

Soln:

$$\vec{F} = 5xy\vec{i} + 2y\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{F}\cdot d\vec{r} = 5xy\,dx + 2y\,dy$$

Given : $y = x^3 \Rightarrow dy = 3x^2\,dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_1^2 5x(x^3)\,dx + 2(x^3)3x^2\,dx$$

$$= \int_1^2 (5x^4 + 6x^5)\,dx$$

$$= \left[\frac{5x^5}{5} + \frac{6x^6}{6}\right]_1^2$$

$$= [32 + 64 - (1+1)]$$

$$= 94$$

(4) Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$ along the line joining the points $(0,0,0)$ to $(2,1,1)$.

Soln:

$$\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (2y+3)\,dx + xz\,dy + (yz-x)\,dz$$

The equation of line joining the points $(0,0,0)$ & $(2,1,1)$ is

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$$x = 2t, \quad y = t, \quad z = t$$

$$\vec{F}\cdot d\vec{r} = 2(2t+3)\,dt + 2t^2\,dt + (t^2 - 2t)\,dt$$

$$= (3t^2 + 2t + 6)\,dt$$

At $x=0, y=0, z=0 \Rightarrow t=0$

At $x=2, y=1, z=1 \Rightarrow t=1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2 + 2t + 6)\,dt$$

$$= \left[t^3 + t^2 + 6t\right]_0^1$$

$$= 8$$

(5) If $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x=t,\ y=t^2,\ z=t^3$

Soln:

$\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\vec{F}\cdot d\vec{r} = (3x^2 + 6y)dx - 14yz\,dy + 20xz^2\,dz$

Given: $x = t,\ y = t^2,\ z = t^3$

$\Rightarrow dx = dt,\ dy = 2t\,dt,\ dz = 3t^2\,dt$

$\vec{F}\cdot d\vec{r} = (3t^2 + 6t^2)dt - 14(t^2 t^3)2t\,dt + 20(t\cdot t^6)3t^2\,dt$

$= (9t^2 - 28t^6 + 60t^9)\,dt$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= 5$

SURFACE INTEGRAL:

Let S be a surface whose projection R_{xy} on the xy plane is such that the points on S have a 1–1 correspondence with the points on R_{xy}. Let ds be a vector element of the area. Then

$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xy}} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

For yz plane, $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{yz}} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

For xz plane, $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xz}} \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$

The surface integral $\iint_S \vec{F}\cdot d\vec{s}$ represents the total flux of $\vec{F}$ through the whole surface.

① Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\vec{i} + x\vec{j} + 3y^2z\vec{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ & $z=5$.

Soln:

$$\vec{F} = z\vec{i} + x\vec{j} + 3y^2z\vec{k}$$

$$\varphi = x^2+y^2-16$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j}$$

Now $$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}}$$

$$= \frac{x\vec{i}+y\vec{j}}{4} \quad (\because x^2+y^2=16).$$

Let us consider yz-plane

z varies from 0 to 5

y varies from 0 to 4

$x^2+y^2=16$, Put $x=0$, $y^2=16 \Rightarrow y=4$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{yz}} \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$$

$$= \int_0^5\int_0^4 (z\vec{i}+x\vec{j}+3y^2z\vec{k})\cdot\frac{(x\vec{i}+y\vec{j})}{4}\cdot\frac{dy\,dz}{\left|\frac{x\vec{i}+y\vec{j}}{4}\cdot\vec{i}\right|}$$

$$= \int_0^5\int_0^4 \left(\frac{zx}{4}+\frac{xy}{4}\right)\frac{dy\,dz}{\frac{x}{4}}$$

$$= \int_0^5\int_0^4 \cancel{\frac{x}{4}}\,(z+y)\,\frac{dy\,dz}{\cancel{x/4}}$$