Divergence of a vector point function

Let $\vec{F}$ be any given continuously differentiable point function then the divergence of $\vec{F}$ is defined as $\vec{F} = \nabla \cdot \vec{F} = \vec{i}\frac{\partial F_1}{\partial x} + \vec{j}\frac{\partial F_2}{\partial y} + \vec{k}\frac{\partial F_3}{\partial z}$

Solenoidal of a vector

Let a vector $\vec{F}$ is said to be solenoidal if $\nabla \cdot \vec{F} = 0$.

Curl of a vector point function

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function then the curl or rotation of $\vec{F}$ is defined by

$$\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F_1} & \vec{F_2} & \vec{F_3} \end{vmatrix}$$

Irrotational vector

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$

Problem 1

Show that $\vec{F} = \underbrace{3y^4z^2}_{F_1}\vec{i} + \underbrace{4x^3z^2}_{F_2}\vec{j} - \underbrace{3x^2y^2}_{F_3}\vec{k}$ is solenoidal

$$\nabla \cdot \vec{F} = \vec{i}\,\frac{\partial F_1}{\partial x} + \vec{j}\,\frac{\partial F_2}{\partial y} + \vec{k}\,\frac{\partial F_3}{\partial z}$$

$$= \vec{i}\,\frac{\partial}{\partial x}(3y^4z^2) + \vec{j}\,\frac{\partial}{\partial y}(4x^3z) + \vec{k}\,\frac{\partial}{\partial z}(-3x^2y^2)$$

$$\nabla \cdot \vec{F} = 0$$

$\therefore$ The given $\vec{F}$ is solenoidal.

Problem 2

Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F_1} & \vec{F_2} & \vec{F_3} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \vec{i}(x - x) - \vec{j}(y - y) + \vec{k}(z - z)$$

$$= 0$$

$\therefore$ $\vec{F}$ is irrotational.

Problem 3

Find a such that $(3x-2y+z)\vec{i}+(4x+ay-z)\vec{j}+(x-y+2z)\vec{k}$ is solenoidal.

$$\nabla\cdot\vec{F} = \vec{i}\,\frac{\partial F_1}{\partial x} + \vec{j}\,\frac{\partial F_2}{\partial y} + \vec{k}\,\frac{\partial F_3}{\partial z}$$

$$= \vec{i}\,\frac{\partial}{\partial x}(3x-2y+z) + \vec{j}\,\frac{\partial}{\partial y}(4x+ay-z) + \vec{k}\,\frac{\partial}{\partial z}(x-y+2z)$$

$$\nabla\cdot\vec{F} = 3\vec{i}+a\vec{j}+2\vec{k}$$

$$0\cdot(\vec{i}+\vec{j}+\vec{k}) = (3\vec{i}+a\vec{j}+2\vec{k})\cdot(\vec{i}+\vec{j}+\vec{k})$$

$$0 = 3+a+2$$

$$a = -5$$

Problem 4

If $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ find the constant a, b, c if $\vec{F}$ is irrotational.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(F_3) - \frac{\partial}{\partial z}(F_2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(F_3) - \frac{\partial}{\partial z}(F_1)\right] + \vec{k}\left[\frac{\partial}{\partial x}(F_2) - \frac{\partial}{\partial y}(F_1)\right]$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] + \vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$0 = \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$$

$c+1=0$ $\quad$ $-4+a=0$ $\quad$ $b-2=0$

$c=-1$ $\quad$ $a=4$ $\quad$ $b=2$.