



Example: Verify green theorem for  $\int_C (xy + y^2) dx + x^2 dy$  where  $C$  is a closed region bounded by  $y = x^2$ ,  $y = 2x$ .

Soln:



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UNIT-I Vector calculus

Green's theorem

Given,  $\int_C (xy + y^2) dx + x dy$

Let  $y = x^2$  and  $x = y$ . The region R is bounded by  $x=0$ ,  $x=1$ ,  $y=x^2$ , and  $y=x$ .

Sub  $y=x$  in  $y=x^2$ :  
 $x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x=0 \text{ or } x=1$

If  $x=0$  then  $y=0$   
If  $x=1$  then  $y=1$

RHS :  $= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$= \int_0^1 \int_{x^2}^x (2x - x - 2y) dx dy$

$= \int_0^1 \int_{x^2}^x (x - 2y) dy dx$

$= \int_0^1 \left[ \int_{x^2}^x x dy - \int_{x^2}^x 2y dy \right] dx$

$= \int_0^1 \left[ (xy) \Big|_{x^2}^x - 2 \left[ \frac{y^2}{2} \right] \Big|_{x^2}^x \right] dx$

$= \int_0^1 \left[ (x^2 - x^3) - 2 \left[ \frac{x^2 - x^4}{2} \right] \right] dx$

$= \int_0^1 \left[ (x^2 - x^3) - [x^2 - x^4] \right] dx$

$= \int_0^1 (x^4 - x^3) dx$



$$\begin{aligned}
 &= \int_0^1 x^4 dx - \int_0^1 x^3 dx \\
 &= \left[ \frac{x^5}{5} \right]_0^1 - \left[ \frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20} \\
 \text{LHS : } &\int_C (xy + y^2) dx + x^2 dy = \int_{OA} (xy + y^2) dx + x^2 dy + \\
 &\quad \int_{AO} (xy + y^2) dx + x^2 dy \\
 \text{Along OA, } &x = y, \quad \frac{dy}{dx} = 1 \\
 &dx = dy \\
 &dy = dx \\
 &= \int_0^1 (xx^2 + x^4) dx + x^2 (2x) dx \\
 &= \int_0^1 (x^3 + x^4) dx + 2x^3 dx \\
 &= \int_0^1 (x^3 + x^4) dx + \int_0^1 2x^3 dx \\
 &= \left[ \frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 + \left[ \frac{2x^4}{4} \right]_0^1 \\
 &= \left[ \frac{1}{4} + \frac{1}{5} \right] + \frac{1}{2} \\
 &= \frac{9}{20} + \frac{1}{2} = \frac{19}{20} \\
 \text{Along AO, } &x = y, \quad \frac{dy}{dx} = 1 \\
 &dx = dy
 \end{aligned}$$



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$$\begin{aligned}
 & \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy = \int_0^b \left[ xy + \frac{y^3}{3} \right] dx = pb^2 + \frac{b^3}{3} \\
 & = \int_0^b (x^2 + x^2) dx = \int_0^b (2x^2) dx = \frac{2}{3} b^3 \\
 & = \int_0^b 3x^2 dx = \frac{1}{3} b^3 \\
 & \therefore \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy = pb^2 + \frac{b^3}{3} (p + \frac{b^2}{3}) \quad (LHS) \\
 & \int_{AO} (xy + y^2) dx + \int_{OB} x^2 dy = \int_{OA} (xy + y^2) dx + \int_{OB} x^2 dy \\
 & \text{Now, } \int_{OA} (xy + y^2) dx + \int_{OB} x^2 dy = \int_0^b (xy + y^2) dx + \int_0^b x^2 dy \\
 & = \frac{19}{20} b^3 + (-1) \int_0^b xy dx = pb^2 \\
 & = \frac{19}{20} b^3 - pb^2 = \frac{-1}{20} (pb^2 - pb^2) \\
 & \therefore LHS = RHS \quad \therefore \text{Hence proved.}
 \end{aligned}$$



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UNIT-I Vector calculus

Green's theorem