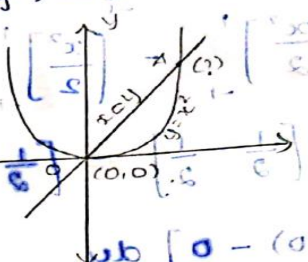
Example:

Verify green theorem for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed region bounded by $y = x^2$, $y = x$

Soln:

Given,

$$\int_C (xy + y^2)\, dx + x^2 dy.$$



$y = x^2 \rightarrow (1)$

$y = x \rightarrow (2)$.

sub $y = x$ in (1)

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ $x - 1 = 0$

$x = 0$ $x = 1$

If $x = 0$ then $y = 0$

If $x = 1$ then $y = 1$

$\frac{\partial M}{\partial y} = 2y + x$ $\frac{\partial N}{\partial x} = 2x$

RHS : $$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dx\, dy.$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dy\, dx.$$

$$= \int_0^1 \left[\int_{x^2}^{x} x\, dy - \int_{x^2}^{x} 2y\, dy\right] dx$$

$$= \int_0^1 \left[(xy)_{x^2}^{x} - 2\left[\frac{y^2}{2}\right]_{x^2}^{x}\right] dx.$$

$$= \int_0^1 \left(x^2 - x^3 - 2\left[\frac{x^2}{2} - \frac{x^4}{2}\right]\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 - 2\left[\frac{x^2 - x^4}{2}\right]\right] dx$$

$$= \int_0^1 (x^2 - x^3 - x^2 + x^4)\, dx.$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \int_0^1 x^4\, dx - \int_0^1 x^3\, dx$$

$$= \left(\frac{x^5}{5}\right)_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20}$$

$$= \frac{-1}{20}.$$

LHS:

$$\int_C (xy+y^2)dx + x^2 dy = \int_{OA} (xy+y^2)\, dx + x^2 dy + \int_{AO} (xy+y^2)\, dx + x^2 dy$$

Along OA, $x^2 = y$

$$2x = \frac{dy}{dx}$$

$$dy = 2x\, dx$$

$$= \int_0^1 (x x^2 + x^4)\, dx + x^2 (2x) dx$$

$$= \int_0^1 (x^3 + x^4)\, dx + 2x^3 dx$$

$$= \int_0^1 (x^3 + x^4)\, dx + \int_0^1 2x^3\, dx$$

$$= \left[\left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{2x^4}{4}\right]_0^1\right]$$

$$= \left[\frac{1}{4} + \frac{1}{5}\right] + \frac{1}{2}$$

$$= \frac{9}{20} + \frac{1}{2} = 19/20.$$

Along AO, $x = y$

$$1 = \frac{dy}{dx}$$

$$dx = dy$$

$$\int_{AO} (xy + y^2)\, dx + x^2 dy$$

$$= \int_1^0 (x^2 + x^2)\, dx + x^2 dx.$$

$$= \int_1^0 (x^2 + x^2 + x^2)\, dx.$$

$$= \int_1^0 3x^2\, dx.$$

$$= \left[\frac{3x^3}{3}\right]_1^0$$

$$= -1$$

Now,

$$LHS = \int (xy + y^2)\, dx + x^2 dy + \int (xy + y^2)\, dx + x^2 dy$$

$$= \frac{19}{20} + (-1)$$

$$= \frac{19}{20} - 1 = -1/20$$

$\therefore$ LHS = RHS

$\therefore$ Hence proved.