



# SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT-I Vector calculus

Gauss Divergence theorem

Gauss divergence Theorem:  
 If  $\nabla$  is the volume enclosed by a closed surface  $S$  and if a vector function  $\vec{F}$  is continuous and has continuous partial derivatives in  $V$  on  $S$ , then,

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV.$$

Example :

Verify Gauss Divergence theorem for  $\vec{F} = (x^2 - y^2) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k}$ , where  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ ,  $0 \leq z \leq c$ .

Soln: Given,

$$\vec{F} = (x^2 - y^2) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k}$$

RHS:

$$\nabla \cdot \vec{F} = \left( \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot ((x^2 - y^2) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k})$$

$$= \frac{\partial}{\partial x} (x^2 - y^2) + \frac{\partial}{\partial y} (y^2 - zx) + \frac{\partial}{\partial z} (z^2 - xy)$$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z.$$

$$\iiint_V \nabla \cdot \vec{F} dV = 2 \int_0^c \int_0^b \int_0^a (2x + 2y + 2z) dx dy dz$$

$$= 2 \int_0^c \int_0^b \left[ \frac{x^2}{2} + xy + xz \right]_0^a dy dz$$

$$= 2 \int_0^c \left[ \int_0^b \left[ \frac{a^2}{2} + ay + az \right] dy \right] dz$$

$$= 2 \int_0^c \left[ \frac{a^2}{2} y + \frac{ay^2}{2} + azy \right]_0^b dz$$



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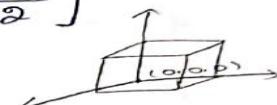
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$$\begin{aligned}
 &= 2 \int_0^c \left[ \frac{a^2}{2} b + \frac{ab^2}{2} + abc \right] dz \\
 &= 2 \left[ \frac{a^2 b}{2} z + \frac{ab^2}{2} z + \frac{abc z^2}{2} \right]_0^c \\
 &= 2 \left[ \frac{a^2 b c}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right] \\
 &= 2abc \left[ \frac{(a+b+c)}{2} \right] \\
 &= abc(a+b+c)
 \end{aligned}$$



LHS:  $\iint_S \vec{F} \cdot \hat{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n} ds + \iint_{S_2} \vec{F} \cdot \hat{n} ds + \dots + \iint_{S_6} \vec{F} \cdot \hat{n} ds$

surfaces	equation	$\hat{n}$	$ds$
$S_1$	$x=0$	$-\vec{i}$	$dydz$
$S_2$	$x=a$	$\vec{i}$	$dydz$
$S_3$	$y=0$	$-\vec{j}$	$dxdz$
$S_4$	$y=b$	$\vec{j}$	$dxdz$
$S_5$	$z=0$	$-\vec{k}$	$dxdy$
$S_6$	$z=c$	$\vec{k}$	$dxdy$

On  $S_1$ :

$$\begin{aligned}
 \iint_{S_1} \vec{F} \cdot \hat{n} ds &= - \int_0^c \int_0^b (x^2 - yz) dy dz \\
 &= - \int_0^c \int_0^b -yz dy dz \\
 &= \int_0^c \left[ \frac{y^2}{2} \right]_0^b dz \\
 &= \int_0^c \left[ \frac{b^2}{2} \right] z dz \\
 &= \frac{b^2}{2} \cdot \left[ \frac{z^2}{2} \right]_0^c \\
 &= \frac{b^2}{2} \times \frac{c^2}{2} = \frac{(bc)^2}{4}
 \end{aligned}$$



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On  $\delta_2$ ,  $x=a$ ,  $\vec{A} = \vec{j}$   $ds = dy dz$ .

$$\begin{aligned}\iint_{\delta_2} \vec{F} \cdot \hat{n} ds &= \int_0^c \left[ \int_0^b (cx^2 - yz) dy dz \right] dx \\ &= \int_0^c \left[ \int_0^b (a^2 - yz) dy \right] dz \\ &= \int_0^c \left[ a^2y - \frac{yz^2}{2} \right]_0^b dz \\ &= \int_0^c \left[ a^2b - \frac{b^2z}{2} \right] dz \\ &= \left[ a^2bz - \frac{b^2z^2}{2} \right]_0^c \\ &= a^2bc - \frac{(bc)^2}{4}\end{aligned}$$

On  $\delta_3$

$$\begin{aligned}\iint_{\delta_3} \vec{F} \cdot \hat{n} ds &= \int_0^c \left[ \int_0^a (cy^2 - zx) dx dz \right] dy \\ &= \int_0^c \left[ \int_0^a zx dx \right] dz \\ &= \int_0^c \left[ \frac{zx^2}{2} \right]_0^a dz \\ &= \int_0^c \frac{xa^2}{2} dz \\ &= \left[ \frac{a^2}{2} \times \frac{z^2}{2} \right]_0^c \Rightarrow \frac{(ac)^2}{4}\end{aligned}$$

On  $\delta_4$ :

$$\begin{aligned}\iint_{\delta_4} \vec{F} \cdot \hat{n} ds &= \int_0^c \left[ \int_0^a (cy^2 - zx) dy dz \right] dx \\ &= \int_0^c \left[ \int_0^a (y^2 - zx) dy \right] dz \\ &= \int_0^c \left[ \int_0^a (b^2 - zx) dx \right] dz\end{aligned}$$



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$$\begin{aligned}
 &= \int_0^c \left[ b^2 z - \frac{z^2 x^2}{2} \right]_0^a dz \\
 &= \int_0^c \left[ ab^2 z - \frac{z^2 x^2}{2} \right] dz \\
 &= \left[ ab^2 z - \frac{b^2 x^2 z^2}{2} \right]_0^c \\
 &= \left[ ab^2 c - \frac{a^2 x^2 c^2}{2} \right] \\
 &= ab^2 c - \frac{(ac)^2}{4}
 \end{aligned}$$

On  $\delta 5$

$$\begin{aligned}
 \iint_{\delta 5} \vec{F} \cdot \hat{n} ds &= \int_0^b \int_0^a (z^2 - xy) dx dy \\
 &= \int_0^b \int_0^a xy dx dy \\
 &= \int_0^b \left[ \frac{x^2 y}{2} \right]_0^a dy \\
 &\text{substituting } y = \frac{a^2}{x} \Rightarrow \int_0^b \frac{a^2}{2} dy
 \end{aligned}$$

$$\begin{aligned}
 &\text{substituting } y = \frac{a^2}{x} \Rightarrow \int_0^b \frac{a^2}{2} \left[ \frac{y^2}{2} \right]_0^a dy \Rightarrow \frac{a^2}{2} \times \frac{b^2}{2} \\
 &= \frac{(ab)^2}{4}
 \end{aligned}$$

On  $\delta 6$

$$\begin{aligned}
 \iint_{\delta 6} \vec{F} \cdot \hat{n} ds &= \int_0^b \int_0^a (z^2 - xy) dx dy \\
 &= \int_0^b \left[ z^2 x - \frac{x^2 y}{2} \right]_0^a dy \\
 &= \int_0^b \left( ca^2 - \frac{a^2 y}{2} \right) dy \\
 &= \int_0^b \left( c^2 a y - \frac{a^2 y^2}{2} \right)_0^b dy
 \end{aligned}$$



$$\begin{aligned} &= abc^2 - \frac{a^2 b^2}{4} \\ \iint_S \vec{F} \cdot \hat{n} \, d\sigma &= \frac{(abc)^2}{4} + \frac{a^2 bc}{4} - \frac{(bc)^2}{4} + \frac{(ac)^2}{4} + \\ &\quad \frac{ab^2 c}{4} - \frac{(a^2 c)^2}{4} + \frac{(ab)^2}{4} + \frac{abc^2}{4} \\ &= \frac{(ab)^2}{4} \\ &= a^2 bc + ab^2 c + abc^2 \\ &= abc [a + b + c] \end{aligned}$$