Gauss divergence Theorem:

If V is the volume enclosed by a closed surface S and If a vector function $\vec{F}$ is continous and It has continous partial derivatives in V on S then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

Example:

Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$, where $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Soln: Given,

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

RHS:

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left((x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$\nabla\cdot\vec{F} = 2x + 2y + 2z.$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = 2\int_0^c\int_0^b\left[\int_0^a (x+y+z)\,dx\right]dy\,dz$$

$$= 2\int_0^c\int_0^b\left[\frac{x^2}{2} + xy + zx\right]_0^a dy\,dz$$

$$= 2\int_0^c\left[\int_0^b\left[\frac{a^2}{2} + ay + az\right]dy\right]dz$$

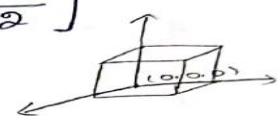
$$= 2\int_0^c\left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right]_0^b dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}b + \frac{ab^2}{2} + azb\right] dz$$

$$= 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= 2abc\left[\frac{(a+b+c)}{2}\right]$$



$$= abc(a+b+c)$$

LHS: $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds + \dots + \iint_{S_6} \vec{F}\cdot\hat{n}\, ds$

surfaces	equation	$\hat{n}$	ds
S_1	$x = 0$	$-\vec{i}$	$dy\,dz$
S_2	$x = a$	$\vec{i}$	$dy\,dz$
S_3	$y = 0$	$-\vec{j}$	$dx\,dz$
S_4	$y = b$	$\vec{j}$	$dx\,dz$
S_5	$y = 0$	$-\vec{k}$	$dx\,dy$
S_6	$z = c$	$\vec{k}$	$dx\,dy$.

On S_1.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = -\int_0^c\int_0^b (x^2 - yz)\, dy\, dz$$

$$= -\int_0^c\int_0^b -yz\, dy\, dz$$

$$= \int_0^c \left[\frac{y^2}{2}\right]_0^b z\, dz$$

$$= \int_0^c \left[\frac{b^2}{2}\right] z\, dz$$

$$= \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2}{2}\times\frac{c^2}{2} = \frac{(bc)^2}{4}$$

On S_2, $x = a$, $\hat{n} = \vec{j}$, $ds = dy\,dz$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c \int_0^b (x^2 - yz)\,dy\,dz$$

$$= \int_0^c \left[\int_0^b (a^2 - yz)\,dy\right] dz$$

$$= \int_0^c \left[a^2 y - \frac{y^2 z}{2}\right]_0^b dz$$

$$= \int_0^c \left[a^2 b - \frac{b^2 z}{2}\right] dz$$

$$= \left[a^2 bz - \frac{b^2}{2}\times\frac{z^2}{2}\right]_0^c$$

$$= a^2 bc - \frac{(bc)^2}{4}$$

On S_3

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c \int_0^a (y^2 - zx)\,dx\,dz$$

$$= \int_0^c \int_0^a zx\,dx\,dz$$

$$= \int_0^c \left[\frac{zx^2}{2}\right]_0^a dz$$

$$= \int_0^c \frac{za^2}{2}\,dz$$

$$= \int_0^c \frac{a^2}{2}\,z\,dz$$

$$= \left[\frac{a^2}{2}\times\frac{z^2}{2}\right]_0^c \Rightarrow \frac{(ac)^2}{4}$$

On S_4 :

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c \int_0^a (y^2 - zx)\,dy\,z\,dz.$$

$$= \int_0^c \int_0^a (y^2 - zx)\,dx\,dz$$

$$= \int_0^c \int_0^a (b^2 - zx)\,dx\,dz$$

$$= \int_0^c \left[b^2 x - \frac{z x^2}{2} \right]_0^a dz$$

$$= \int_0^c \left[ab^2 - \frac{z a^2}{2} \right] dz$$

$$= \left[ab^2 z - \frac{a^2}{2} \frac{z^2}{2} \right]_0^c$$

$$= \left[ab^2 c - \frac{a^2}{2} \times \frac{c^2}{2} \right]$$

$$= ab^2 c - \frac{(ac)^2}{4}$$

On S_5

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a (z^2 - xy) \, dx \, dy .$$

$$= \int_0^b \int_0^a xy \, dx \, dy$$

$$= \int_0^b \left[\frac{x^2}{2} y \right]_0^a dy$$

$$= \int_0^b \frac{a^2}{2} y \, dy$$

$$= \frac{a^2}{2} \left[\frac{y^2}{2} \right]_0^b = \frac{a^2}{2} \times \frac{b^2}{2}$$

$$= \frac{(ab)^2}{4}$$

On S_6

$$\iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a (z^2 - xy) \, dx \, dy$$

$$= \int_0^b \left[z^2 x - \frac{x^2}{2} y \right]_0^a dy$$

$$= \int_0^b \left(c^2 a - \frac{a^2 y}{2} \right) dy$$

$$= \left(c^2 a y - \frac{a^2 y^2}{2 \times 2} \right)_0^b$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \frac{\cancel{(bc)^2}}{\cancel{4}} + \frac{a^2bc}{1} - \frac{\cancel{(bc)^2}}{\cancel{4}} + \frac{\cancel{(ac)^2}}{\cancel{4}} + \frac{ab^2c}{1} - \frac{\cancel{(ac)^2}}{\cancel{4}} + \frac{\cancel{(ab)^2}}{\cancel{4}} + \frac{abc^2}{1} - \frac{\cancel{(ab)^2}}{\cancel{4}}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,[a+b+c]$$